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THE QUANTIZATION OF GEOMETRY

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I N S T I T U T E   O F   F I E L D   P H Y S I C S

DEPARTMENT OF PHYSICS

University of North Carolina at Chapel Hill

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List of Errata

p. 5, line 9,	Replace "coveriant" by "covariant."
p. 8, line 15,	Replace "... system. This system ..." by " ... system, which ..."
p. 11, line 2,	Replace "golloWing" by "following."
p. 16, line 2,	The symbol on this line is "$\delta\phi^{i}$."
p. 19, line 7,	Replace "dynamical" by "canonical."
p. 20, line 20,	Replace "Let us consider ..." by "Let us therefore consider..."
line 22,	Replace "It will be shown ..." by "The Uncertainty Principle then states..."
p. 21,	Sentence beginning at bottom of page should be changed to read "It will then be shown to hold also for the system S and by judicious choice of apparatuses and couplings, may therefore be extended to all physical systems."
p. 25,	The tilde "~" should be inserted in Eqs. (9.2.29) and (9.2.30).
p. 27, line 9,	Replace "mathe -" by "mathe •".
line 14,	Replace "physics" by "formalism."
p. 30, line 16,	Replace "true null eigenvectors" by "true eigenvectors corresponding to zero eigenvalues."
p. 33, line 8,	Replace "true null eigenvectors" by "true eigenvectors corresponding to zero eigenvalues."
p. 36, Eq. (9.3.19),	Replace "$\delta\xi^{\pm}$" by "$\delta\xi^{A}$."
p. 40, Eq. (9.3.37),	Replace "δ_{BA}" by $\delta_{B}{}^{A}$."

p. 42, Eq. (9.3.40),	Replace "$-\delta_{(A,B)}$" by "$-\delta_{\epsilon(A,B)}$."
p. 43, line 15,	Replace "... the chapter." by "... Part I."
p. 46,	Sentence beginning at bottom of page should be changed to read "In Section 7 the extension of the method to the fully relativistic situation in which the gravitational field itself is given dynamical properties will be discussed."
p. 47, line 2,	Omit "only."
p. 48, line 14,	Insert "Φ" after "1-column array."
p. 49, lines 3 and 4,	Replace "$\delta\tau^{\mu'\nu'}$" by "$\delta T^{\mu'\nu'}$."
Eq.(9.3.60),	Insert "T" after integral sign in first line.
p. 50, line 5,	Equation number should read "(9.3.62)."
p. 52, Eq. (9.4.7),	Replace "m" by "m^{-1}."
p. 53, Eq. (9.4.9),	Replace all "=" signs by "≡" signs. All x's should be boldface: "$\boldsymbol{x}$"
Eq. (9.4.10),	Insert " -" sign in front of the lower Poisson bracket in this equation.
line 8,	Replace "$\theta(t - t')$" by "$\dot{\theta}(t - t')$."
p. 55, Eq. (9.4.17),	Replace $\boldsymbol{x}$'s by $\dot{\boldsymbol{x}}$'s.
p. 57,	Lines 19 and 20 should run together, with no new paragraph at bottom of page.
p. 58, line 8,	Replace "... dot. (Note ..." by " ... dot (not ..."
line 19,	Replace final "(" by a comma.
p. 63, line 4,	Replace "$-\delta(\tau - ')$" by "$-\delta(\tau - \tau')$."
p. 64,	Eq. (9.4.81) should read "$\delta S \equiv -\int m(-\dot{x}^2)^{-\frac{1}{2}} x_\mu d(\dot{x}^\mu \delta t) = \int m \delta\dot{\tau}\, d\tau$,"

p. 67, line 11, Replace "one" by "once."

p. 68, line 12, Replace "u" by "$\mathbf{u}$."

p. 70, Insert " $\equiv$ " sign in Eq. (9.5.7).

p. 71, line 2, Replace "δW" by "$\delta \dot{W}$."

Eq. (9.5.16), Replace "=" sign by " $\equiv$ " sign.

p. 72, Eq. (9.5.23) should read "$\dot{f} = (f, H)$."

p. 73, Eq. (9.5.26), Place "(" in front of second line of this equation and bracket second and third lines together with a large curly bracket. The subscripts "J'" in the fourth line should be replaced by "j'" and the first $x_{j'}$ should be dotted.

line 7, Replace "$\delta(t - t')$" by "$\dot{\delta}(t - t')$."

p. 75, line 12, Replace "actual" by "momentary."

line 14, Replace "x_j" by "$\langle x_j \rangle$."

p. 78, line 4, Replace "tha" by "the."

line 17, Replace "l" by "$\mathbf{l}$."

The second of Eqs. (9.5.49) is missing a "l" in the numerator.

p. 80, line 7, Insert comma after "in fact."

Eq. (9.5.58), Replace " $\frac{1}{c_\ell}$ " by " $\frac{1}{2c_\ell}$."

p. 82, line 1, Replace " (9.5.58)" by "(9.5.60)."

p. 83, lines 9 and 10, Replace "strictly local considerations" by "use of local rest frames."

p. 84, line 14, Insert comma after "functions."

p. 86, Eq. (9.5.76) should read

$$c^{\mu\nu\sigma\tau} \equiv (-\dot{x}^2)^{\frac{1}{2}} \frac{\partial(t,u)}{\partial(x)} P^{\mu}{}_{\iota} P^{\nu}{}_{\kappa} P^{\sigma}{}_{\lambda} P^{\kappa}{}_{\rho} \dot{x}^{\iota}{}_{,a} x^{\kappa}{}_{,b} x^{\lambda}{}_{,c} x^{\rho}{}_{,d} c^{abcd}$$

p. 86, line 11, Replace "(9.4.2)" by "(9.4.25)."

p. 88, line 2, Replace "(9.5.52)" by "(9.3.52)."

lines 7 and 8, Replace "... the following section," by "... Part II,"

p. I-iv, Third line below Eq. (A.11): Replace "$P_i{}^A$, Q_{iA}" by "$P_i{}^A$, Q_{iA}."

Fourth line below Eq.(A.11): Replace "$P_i{}^A$, A_{iA}" by "$P_i{}^A$, Q_{iA}."

p. I-vi, Eq. (A.18), Factors following "$\frac{\partial}{\partial x^\mu}$" in integrand of second line of this equation should be enclosed in square brackets. The quantities "$f^\mu{}_{i'j}$" and "$f^\mu{}_{i'j}$" in the square matrix should be interchanged.

p. I-vii a, The lower right hand element of the matrix in the third line of Eq. (A.31) should read

"$P_i{}^A\ Q_{k''A} + P_{k''}{}^A\ Q_{iA}$, "

i.e., with a "+" sign instead of a "-" sign. This equation should be inserted after Eq. (A.30) at the bottom of p. I-vii.

p. I-viii, Eq. (A.34) should be renumbered "(A.33)." A square bracket should be inserted at the end of the third line of this equation.

p. I-ix, Eqs. (A.35) to (A.38) should be renumbered from "(A.34)" to "(A.37) consecutively.

p. I-x, Eq. (A.39) should be renumbered "(A.38)."

p. 89, line 14, Replace "every thing" by "everything."

p. 90, line 11, Replace "qunatities" by "quantities."

p. 96, line 1, Replace "$F^{\epsilon'\zeta'}$ " by "$F_\alpha{}^{\epsilon'\zeta'}$."

line 4, Replace "$F^{\alpha\beta}{}_{J}$ " by "$F^{\alpha\beta}{}_{J'}$."

p. 98, line 1, Insert "τ" after "subscripts."

Eq. (9.6.44), First term on right should read "$G^{\alpha\gamma'}$."

p. 100, line 10, Replace "considerable" by "considerably."

p. 102, line 10, Replace "varifies" by "verifies."

p. 103, Third line from bottom: Replace "Eq." by "Eqs."

p. 106, Second line from bottom: Replace "conditiona" by "conditions."

p. 109, line 8, Replace "If" by "It."

p. 112, First term on right of Eq. (9.7.4) should read

"$R_{\alpha\beta\gamma\delta\cdot\epsilon}\,\delta z^{\epsilon}$"

p. 117, Eq. (9.7.18), Terms following the " $-\frac{1}{2}$ " in the second and third lines of this equation should be enclosed in square brackets.

line 18, Insert "for" after "$D^{\pm}$."

p. 118, Eq. (9.7.21), Indices on "R" should be in lower position.

p. 121, Eq. (9.7.41), Insert "i" after "$\frac{1}{4}$".

p. 122, Eq. (9.7.42), "D" should be dotted.

last line, Replace "$(k^{\mu}) - (\underset{\sim}{k}, k^{0})$" by "$(k^{\mu}) = (k^{0}, \underset{\sim}{k})$."

p. 125, Eqs. (9.7.60), Bracket these equations together with a large curly bracket.

p. 127, line 17, Replace "x'-axis" by "x^{1}-axis."

p. 128, line 16, Replace "elastic moduli c_{abcd}" by "internal stresses."

p. 130, line 8, Replace "nust" by "must."

p. 133, line 16, Replace "empmphasized" by "emphasized."

last line, Replace "measure" by "measured."

p. 135, line 17, Replace "magnitued" by "magnitude."

p. 138, Sentence beginning on line 17 should be changed to read "At the beginning of the interval T, instead of adjusting the internal stresses so as to insure strain rigidity we simply let the elastic moduli all fall abruptly to zero."

p. 143,	Eq. (9.8.44) should read	
	"$\Delta z_{1\tau} = m^{-1}(p_{1\tau}'' - p_{1\tau}')\,\Delta t$."	
p. 146, line 14,	Replace "im" by "in."	
p. 149, line 2,	Replace "strain tensor" by "stress-energy density."	
p. 150, Eqs. (9.8.64),	Replace "$\Delta T_{(0)}{}^{1}$" by "$\Delta T_{(0)1}$."	
p. 152, line 15,	Replace "displacement" by displacements."	

There is no page numbered 163.

p. 164, line 10,	Insert "of" after "(D.15)."
p. 169, line 8,	Replace "region, will in virtue ..." by "region will, in virtue ..."
p. 171, line 12,	Replace "is a theory" by "is basically a theory."
p. 173, Second line from bottom:	Replace "... approach. For, an ..." by "... approach, for an ..."
p. II-ii, Eq. (B.10),	Last term should read "$n\omega v^{\mu}v^{\nu}\delta J$."
p. II-iv, Eq. (B.23),	The symbols "∂" have been omitted from the Jacobian in the bottom element of the column matrix.
p. II-vi, Eq. (B.33),	Last term of second line of this equation should read "$G^{\pm}{}_{\gamma\delta\epsilon'\zeta'}$"
Eq. (B.34),	Quantities "$F_{\alpha\epsilon}$" and "$F^{\alpha\beta}{}_{\epsilon}$" in the first column of first matrix should be replaced by "$F_{\alpha\gamma'}$" and "$F^{\alpha\beta}{}_{\gamma'}$" respectively. First matrix should be preceded by a square bracket.
p. II-vii, Eq. (B.35),	First factor on right should be "v^{α}" instead of "v^{δ}."
Last line,	Replace "ρ" by "ρ_0."
p. II-ix, Eqs. (C.13),	Replace final factor "$(\tau - \tau)$" by "$(\tau - t')$."

p. II-xii, Eq. (D.5), "Σ_{ij}" should be dotted.

p. II-xiii, Eq. (D.8), Replace "$\delta(\delta^{\mu}{}_{\beta\cdot\gamma})/\delta\tau$" in the fourth line of this equation by "$\partial_{cov}(\delta^{\mu}{}_{\beta\cdot\gamma})/\partial\tau$," signifying the covariant proper time derivative of $\delta^{\mu}{}_{\beta\cdot\gamma}$.

p. 180, line 1, Replace "$p^{\mu}{}_{\nu}$" by "$P^{\mu}{}_{\nu}$."

Part I

CHAPTER 9

THE QUANTIZATION OF GEOMETRY

Bryce S. DeWitt

(9.1) Introduction.

The development of fundamental concepts in theoretical physics since 1900 has been very much a story of the epistemological analysis of space and time on the one hand and, within the framework of quantum mathematics, of the notions of observation, measurement, and indeterminism on the other. These two aspects of physical theory have always remained sharply distinguished, in spite of the profound influence that each has exerted on the other and the deep connection which undoubtedly exists between them. This chapter is an attempt to indicate the nature of and ideas involved in the problem of removing this division in physics in a manner which goes beyond the already familiar superposition of the ideas of special relativity upon a quantum framework.

In a restricted technical sense the problem under consideration is referred to as "the quantization of the gravitational field." It should be stated at the outset that there is no experimental motivation for the investigation of this problem whatsoever. The inescapable lessons which Nature has been teaching in the laboratory during the past sixty years, concerning her fundamental symmetries and the group theoretical properties of the mathematical formalisms which describe her, have fallen short of providing a complete synthesis of the observational viewpoint in physics. The motivation for "quantizing the gravitational field" therefore consists solely in the fact that the program is utterly logical. It demands no new hypotheses. It rests completely on the general theory of relativity and on conventional quantum theory, of which the latter has been established as correct beyond all doubt, while the former, although lacking as firm an experimental foundation, has, because of its beauty and powerful viewpoint, deeply influenced better established areas of physics.

One should, of course, not fail to mention certain speculations which have been made from time to time (Pauli, 1956; Klein, 1955, 1957; Landau, 1955; Deser, 1957) concerning the possibility that a quantum-fluctuating gravitational field may remove the divergences in conventional relativistic quantum field theories, by providing a natural "cut-off." We shall have occasion later to discuss such a natural limit. However, there exists as yet absolutely no concrete mathematical evidence either to support or to deny these speculations. A long program of formalism building and calculation is an unavoidable prerequisite. We shall therefore dismiss this problem from discussion and turn to the fundamental considerations which will determine the character of the formalism itself.

The problem of constructing a formalism for quantum gravidynamics has been under study for at least the past dozen years and has proved to be a particularly vexing one. No attempt will be made here to give an historical survey of the work that has been done, although lessons learned from it will constitute an important factor in controlling our method of procedure. The bibliography at the end of the chapter contains a fairly complete list of references to work appearing after 1955. For references to earlier work the reader should consult the article by Bergmann in the proceedings of "The Jubilee of Relativity Theory" (Bergmann, 1956).

The problem may be approached from either of two viewpoints, loosely described as the "flat space-time approach" and "the geometrical approach". In the flat space-time approach, which has been investigated by several authors (Feynman, 1957; Thirring, 1959; Gupta, 1952, 1957; Belinfante, 1957; Birkhoff, 1944; Moshinsky, 1950; Rosenfeld, 1930) the gravitational field is regarded as just one of several known physical fields, describable within the Lorentz-invariant framework of a flat space-time. Its couplings with the other fields are largely determined by experiment together with considerations of simplicity involving the mathematics of spin-2 fields. These couplings lead to a contraction or elongation of "rigid" rods and a retardation or advancement of "standard" clocks, which are independent of the individual characteristics of these instruments, in the proximity of gravitating matter as well as in regions containing strong gravitational radiation.

In the geometrical approach to quantization, on the other hand (which owes so much to the work of Bergmann---see bibliography), the theory of gravitation is regarded in classical Einsteinian terms as a theory of the geometry of space-time, in which rigid rods and standard clocks are themselves regarded as providing the local definition of invariant intervals. Both the geometrical and flat space-time points of view have the same real physical content. However, it has been argued that the flat space-time approach provides more immediate access to the concepts of conventional quantum field theory and allows the techniques of the latter theory to be directly applied to gravitation. While there is merit in this argument, too strong an insistence upon it would constitute a failure to have learned the lessons which special relativity has itself already taught. Just as it is now universally recognized as inconvenient (although possible) to derive the Lorentz-Fitzgerald contraction from relativistic modifications in the force laws between atoms, so it will almost certainly prove inconvenient at some stage to approach space-time geometry, even in the quantum domain, in terms of fluctuations in standard intervals which are the same for all physical devices and hence unobservable. In both cases it is the existence of an underlying invariance group which really controls the interpretation of the formalism. In this chapter the geometrical approach will be firmly adhered to and the invariance group will be placed as much as possible in the foreground.

Unfortunately it is precisely the existence of the coordinate invariance group of general relativity which is responsible for most of the difficulties which have been encountered in attempts to quantize geometry. It may be shown by quite general arguments (Utiyama, 1959) that the existence of such a group always gives rise to constraints which must be satisfied by the "initial data" characterizing individual solutions of the dynamical equations. Although a great concentration of effort has been brought to bear on the problem of constraints, no one has yet found a way to formalize the problem without introducing the canonical fundamentals of a Hamiltonian or quasi-Hamiltonian theory (Dirac, 1958, 1959; Arnowitt, Deser and Misner, 1959, 1960; Anderson, 1958, 1959; Bergmann, 1956, 1958). The canonical approach, however, treats space and time asymmetrically and does not fit comfortably with the invariance group. In certain respects it represents a retreat back to the flat space-time viewpoint--- particularly when asymptotically Minkowskian coordinate conditions at infinity are imposed. Moreover, the overriding need to discover a "reduced Hamiltonian", which the constraint problem imposes, has sometimes led to the extravagant claim that the canonical formalism is essential to the quantization program.[1] The canonical viewpoint represents an endeavor to maintain close contact with familiar parts of quantum theory by casting quantum gravidynamics into conventional language. However, the resulting formalism becomes quite complicated already at an elementary level. Furthermore, it is found to be rather removed from immediate and local invariant physical concepts. The possibility must therefore be considered that the conventional language either asks the wrong questions or else poses them incorrectly.

At this point opinion divides. Some workers feel that however arbitrary the distinction between space and time may be, the conventional language is both necessary and appropriate. Others, including the present author, feel that a language which is manifestly coveriant at every stage is not only desirable but attainable. In the following sections a possible way to develop such a language will be indicated. In this development Hamiltonian ideas are dispensed with entirely and space-time is treated in a completely homogeneous fashion.

A basic tool in what follows is a definition of the classical Poisson bracket by means of Green's functions, which is independent of any definitions of pairs of conjugate variables and which is, in effect, a straightforward extension of a definition originally proposed by Peierls (1952). The point of view will be adopted that Poisson brackets (i.e., commutators) should be defined only between invariants, i.e., quantities which are invariant not only under the group of coordinate transformations but also under any other infinite dimensional transformation groups possessed by the dynamical systems under consideration. This automatically eliminates the need for subsidiary conditions, which have always to be specially tailored to each individual system and which have proved so often bothersome in the past. Furthermore, this approach is in accord with the foundations of the quantum theory as expressed in the general theory of measurement. Real physical measurements can be performed only on group invariant quantities, and the interference between two measurements which, via the Uncertainty Principle, defines the commutator, is most immediately described not in terms of canonically conjugated variables at a given instant, but in terms of the Green's functions which express the laws of propagation of small disturbances and which satisfy certain fundamental reciprocal relations.

In quantum electrodynamics this role of the Green's functions was demonstrated at a very early date in the classic paper of Bohr and Rosenfeld (1933), which made no use of subsidiary conditions, and to which the author of the present chapter is heavily indebted as will be immediately apparent in the sections to follow. This indebtedness may seem in one respect surprising, not, to be sure, because of any present-day diminution in the importance of this classic work, but because its content, as Bohr and Rosenfeld have themselves repeatedly indicated, was guided in every way by the existence of an already developed formalism, whereas here we are trying to "put the cart before the horse"---to develop the formalism itself with the aid of the ideas of the theory of measurability. The reason for this, however, lies in the very nature of the general theory of relativity and of its extremely close kinship in point of view with the conceptual foundations of the quantum theory.[2] Furthermore, having the work of Bohr and Rosenfeld already before us is something quite different from doing the same thing, in ignorance of it, for another, more complicated, system.

Now, there are certain immediate obstacles to carrying out a program along the above lines. The first consists in the fact that in the theory of the pure gravitational field the invariants which come easily to mind (e.g., space-time integrals of scalar densities formed out of the metric and its derivatives) have not so far proved to be useful objects with which to test the theory through "Gedankenmessungen." No one has yet found a way of using these objects to obtain detailed information on the internal dynamics of the field. At the present stage of the theory what appears to be needed, if one insists on maintaining manifest covariance, is a means of constructing <u>local</u> <u>invariants</u>. A possible procedure is to introduce four independent scalars ζ^A, $A = 0, 1, 2, 3$, formed out of the metric tensor and its derivatives and then to use these to define an <u>intrinsic</u> <u>coordinate</u> <u>system</u>. To use such a coordinate system for the purpose of constructing local invariants one picks a set of four numbers η^A and looks at the coordinate mesh defined by the ζ^A at the point where $\zeta^A = \eta^A$ (which will be unique if one is lucky). Any local geometrical quantity referred to the local mesh at this point will then provide a set of one or more local invariants. A sample set is

$$T_A{}^B(\eta) \equiv \int \frac{\partial x^\mu}{\partial \zeta^A} \frac{\partial \zeta^B}{\partial x^\nu} \frac{\partial(\zeta)}{\partial(x)} T_\mu{}^\nu(x)\, \delta(\eta - \zeta(x))\, d^4x\,, \qquad (9.1.1)$$

where $T_\mu{}^\nu$ is any mixed tensor formed out of the metric and its derivatives. The $\partial\zeta^B/\partial x^\nu$ are the gradients of the ζ^A, and $\partial(\zeta)/\partial(x)$ and $\partial x^\mu/\partial\zeta^A$ are respectively the determinant and the components of the inverse of the matrix formed from these gradients. The $T_A{}^B$ are the components of the $T_\mu{}^\nu$ in the intrinsic system, taken at the point where the scalars ζ^A have the values η^A respectively. More complicated geometrical objects may be handled in a similar fashion.

Invariants constructed in this manner have been studied by Komar (1958). (See also Bergmann and Komar, 1960). As scalars he has used the four "eigenvalues" of the Riemann tensor (see Géhéniau and Debever, 1956; Pirani, 1957). The use of these scalars, however, and indeed of almost any conceivable scalars built out of the metric tensor and its derivatives, has a serious defect. Such scalars are functionally independent only in regions of space-time possessing a degree of inhomogeneity and asymmetry sufficient to rule out the applicability there of any of the known exact solutions of Einstein's equations as well as any more general solution satisfying "pure radiation" conditions. The situation is precisely analogous to one which occurs in hydrodynamical theory (see Courant and Friedrichs, 1948) in which, in the case of one dimensional isentropic flow, for example, certain functions of the density and velocity, known as Riemann invariants, can be used to define an "intrinsic" coordinate system, the mesh of which is formed by the "characteristic lines." The intrinsic system can be used to identify space-time points, however, only in complicated flow situations involving interacting waves; it becomes degenerate in precisely the cases of constant flow and so-calle "simple waves."

In order to avoid difficulties of this kind we shall introduce directly into the discussion an additional physical system. This system will serve to furnish us with a reasonably fool-proof set of intrinsic coordinates while at the same time forming a combined physical system with the gravitational field. In principle, any additional system which provides a "useful" set of four scalars will do. Actually, we shall choose the most intuitively obvious system possible, namely, a stiff elastic medium carrying a framework of clocks. Sections 5 and 6 are devoted to the description of this system, which proves to be readily amenable to covariant mathematical analysis. Naturally the physical constitution of the medium as well as of the clocks is not dealt with on an atomic level, but only phenomenologically. The limitations which this imposes on the conclusions of the present chapter will be discussed later.

It might be supposed that the elastic medium with its clocks has merely a technical utility, constituting an otherwise foreign element in the discussion. Such is by no means the case. The role played by the medium in providing a physical coordinate system proves to be a fundamental one, as the measurement theoretical analysis will reveal with particular clarity, and serves to bring the conceptual foundations of both the general theory of relativity and the quantum theory into sharp focus. A clock carrying medium of some kind is needed, if only in limited regions of interest, in order to give an operational meaning to the concept of "space-time geometry" in the first place. One may, to be sure, hope that the introduction of a purely phenomenological medium is only an interim measure, which will be superseded eventually by a comprehensive unified theory of elementary particles and fields, containing its own theory of measurement as well as its own interpretation. It has, in fact, been suggested that such a comprehensive theory might already be achievable within the framework of geometry alone (Misner and Wheeler, 1957). Suffice it to say, however, that present formulations of gravitation theory are very poorly suited indeed to the task of yielding such an outcome.

In the following section (§2) the possibility of bypassing the canonical language is proved through a demonstration of the role of the Uncertainty Principle and the theory of measurement in the definition of the Poisson bracket for an arbitrary system. It is shown in a quite general manner that the quantization of a given system implies also the quantization of any other system to which it can be coupled. By a principle of induction, therefore, the quantum theory must immediately be extended to all physical systems, including the gravitational field. Moreover, the pricise form of the commutator between any two observables is uniquely specified. The properties of the Green's functions which enter naturally into this specification, through their ability to describe the propagation of small disturbances, are studied in Section 3. The existence of infinite dimensional invariance groups is easily taken into account, and the consistency of the Poisson bracket definition is established. Although not essential to the quantization program, nor even to the specification of quantum states, the generator of infinitesimal space-time displacements is derived as an illustration of the general methods. In Section 4 these methods are applied to the free particle, as a familiar example, and to the relativistic clock, which is a basic tool in the theory of the measurement of space-time geometry, as has been emphasized by Wigner (1957) (see also Salecker, 1957, and Møller, 1955) and as will be evident in the present work. After further application of these methods to the elastic medium in Section 5, and to its interaction with the gravitational field and clock framework in Section 6, the gravitational field itself is studied in some detail in Section 7. The problem of finding the generator of infinitesimal displacements with respect to the intrinsic coordinate system provided by the elastic medium together with its clock framework is posed in terms of variations in the action functional, and the difficulties involved in solving the problem are explicitly shown. The significance and range of validity of the "weak-field" approximation is examined and the importance of the Riemann tensor as an approximate invariant is emphasized. Graviton spin and polarization states are defined in terms of the Fourier decomposition of the linearized Riemann tensor, and the commutators of the weak-field theory are given. Section 8 is devoted to a study of the question of the actual measurability of the gravitational field in the quantum domain, gollowing closely the arguments of Bohr and Rosenfeld for the electromagnetic field. The measurability is verified to lowest order of perturbation theory, and the statistical predictions of the weak-field theory are confirmed, provided conceptual test bodies of "Bohrian" delicacy are permitted. The analysis, however, must be extended to include an examination of the stresses in the test bodies, as well as in the various compensation mechanisms and momentum-measuring projectiles (photons) which are used, problems which Bohr and Rosenfeld could ignore. In this extension the fundamental length of quantum gravidynamics (see below) makes a repeated appearance as a lower bound on the size of allowable measurement domains, from which it is necessary to draw the conclusion that the very concept of "field strength" can have no objective classical meaning for domains smaller than this, even if any meaning is in fact left to it at such a microscopic level after the limitations imposed by the observed scheme of known elementary particles are taken into consideration. Finally, in Section 9, the author expresses his views on the outlook for the future of the quantum theory of geometry.

The whole chapter is divided into two parts, each having technical appendices at the end. Units are employed for which $\hbar = c = 16\pi G = 1$, where G is the gravitation constant. All quantities are thereby reduced to dimensionless numbers. In these units the masses of the familiar elementary particles lie in the numerical range 10^{-22} to 10^{-18} while the units of length and time are equal to 1.144×10^{-32} cm. and 3.82×10^{-43} sec. respectively. Attention should be called to the following points of notation: The signature of space-time will be taken as $-+++$. The Riemann and Ricci tensors will be taken in the forms

$$R_{\mu\nu\sigma}{}^{\tau} \equiv \Gamma_{\sigma\nu}{}^{\tau}{}_{,\mu} - \Gamma_{\sigma\mu}{}^{\tau}{}_{,\nu} + \Gamma_{\sigma\nu}{}^{\rho}\Gamma_{\rho\mu}{}^{\tau} - \Gamma_{\sigma\mu}{}^{\rho}\Gamma_{\rho\nu}{}^{\tau} , \tag{9.1.2}$$

$$R_{\mu\nu} \equiv R_{\mu\sigma\nu}{}^{\sigma} , \qquad R \equiv R_{\mu}{}^{\mu} , \tag{9.1.3}$$

where $\Gamma_{\mu\nu}{}^{\sigma}$ is the affinity and the comma denotes the ordinary derivative. The covariant derivative will be indicated by a dot.

(9.2) The role of the Uncertainty Principle and the theory of measurement in the definition of the Poisson bracket.

We begin by considering a general physical system describable by a set of localized real dynamical variables ϕ^i. These variables will be functions of one or more continuous parameters, or "coordinates." For definiteness we may regard them as functions of four space-time coordinates x^μ. Everything we say, however, will be equally applicable to systems with either more or fewer parameters, in particular to systems having only a finite number of degrees of freedom, with "time" as the single parameter. Different points of space-time will be distinguished by means of primes: x, x', x'', etc. For compactness the point at which a given variable, such as ϕ^i, is evaluated will be indicated by affixing primes to the index appearing on the variable: e.g., $\phi^{i''}$. For economy in the use of primes the symbol z will also sometimes be used in place of x to designate a point in space-time. Lower case Latin indices from the beginning of the alphabet (a, b, c ...) will always be associated with the symbol z, while those from the middle of the alphabet (i, j, k ...) will be associated with the symbol x.

The dynamical properties of the system will be specified by an <u>action functional</u> S. For our purposes this functional may be regarded as a purely formal expression, to be used to fix the form of the dynamical equations and to determine their transformation properties. For systems with "localized action" S appears as an integral, over all space-time, of any one of a number of equivalent functions of the ϕ^i

(and their derivatives up to some finite order) which differ from one another by total divergences. Questions of the convergence or divergence of this integral are irrelevant[3] (although they are not irrelevant for its variations) and the dynamical equations themselves may without ambiguity be written in the form

$$S_{,i} = 0 \quad , \tag{9.2.1}$$

where the comma followed by an index is here used to denote the <u>variational</u> or <u>functional derivative</u> with respect to ϕ^i at a point. Furthermore, it does not generally matter how many variables ϕ^i are used to describe the system, as long as all descriptions are equivalent. Some of the ϕ^i may, through the dynamical equations, be expressible in terms of derivatives of others, for example.

Even when the minimum possible number of variables is chosen in the action functional, it does not necessarily follow that any one of them is physically measurable when taken by itself. It will often happen that a continuous range of values for the ϕ^i corresponds to one and the same physical situation and hence that these values cannot be physically distinguished. Changes from one set of values to another in the given range are brought about by a set of transformations forming an invariance group for the system, which expresses certain symmetry properties possessed by the system. In the case of infinite dimensional invariance groups, which will be our main concern here, an infinitesimal transformation belonging to the group produces a variation in the ϕ^i having the general form[4]

$$\delta\phi^i = \int R^i{}_{L'}\, \delta\xi^{L'}\, d^4x' \quad , \tag{9.2.2}$$

where the $\delta\xi^L$ are arbitrary infinitesimal functions known as group parameters. Here, capital Latin indices from the middle of the alphabet (L, M, N...) will be associated with the symbol x, while those from the beginning of the alphabet (A, B, C...) will be associated with the symbol z.

The representation of the group which the variables ϕ^i provide, through Eq. (9.2.2), need not be linear but may be quite general. The only restriction on it is the identity

$$\int (R^i{}_{A,j'} R^{j'}{}_{B'} - R^i{}_{B',j'} R^{j'}{}_{A}) d^4x' = \int R^i{}_{L'} c^{L'}{}_{AB'} d^4x' , \tag{9.2.3}$$

where the $c^L{}_{A\,B'}$ are the structure constants of the group, which in turn satisfy the identity

$$\int (c^L{}_{AM'} c^{M'}{}_{B'C''} + c^L{}_{B'M'} c^{M'}{}_{C''A} + c^L{}_{C''M'} c^{M'}{}_{AB'}) d^4x' = 0. \tag{9.2.4}$$

Typically $R^i{}_{L'}$ will be a "differential operator" --- that is, a linear combination of the delta function and its derivatives with coefficients involving the ϕ^i and their derivatives up to some finite order.

The invariance of the physical situation under the transformation (9.2.2) is assured if the action functional remains invariant under it. A group invariant I is evidently characterized by the condition

$$\int I_{,i} R^i{}_A \, d^4x = 0 . \tag{9.2.5}$$

The action S, in particular, must satisfy this condition independently of the dynamical equations.[5] This means that the dynamical equations themselves are not all independent of one another. As has been mentioned in the Introduction such a situation is always associated, in the canonical formalism, with the problem of constraints. It should be pointed out, however, that a functional relationship between the dynamical equations exists only in the case of infinite dimensional groups. In the case of finite dimensional groups the integral is eliminated from Eq. (9.2.2), and the $\delta\phi^i$ cannot generally be made to vanish in remote regions. Therefore the integration by parts which always enters in the derivation of Eq. (9.2.5) cannot be performed, and a total divergence must be added to the integrand of this equation. Instead of encountering a constraint problem, one is thereby led to a conservation law which holds when the dynamical equations are satisfied (Noether, 1918). Beyond this the effect of finite dimensional invariance groups is limited to insuring covariance of the dynamical equations, that is, their invariance in form under the transformations of the group.

By taking the variational derivative of Eq. (9.2.5), with I replaced by S, it is easy to show that under the group transformation (9.2.2) the dynamical equations (9.2.1) are replaced by linear combinations of themselves. We have

$$\begin{aligned} \delta S_{,i} &= \int d^4x' \int d^4z \, S_{,ij'} \, R^{j'}{}_{A} \, \delta\xi^A \\ &= -\int d^4x' \int d^4z \, S_{,j'} \, R^{j'}{}_{A,i} \, \delta\xi^A . \end{aligned} \qquad (9.2.6)$$

Since this relation must hold independently of the particular solution of the dynamical equations which is involved, the change (9.2.2) in the dynamical variables is physically unobservable, at least within the framework of the system S itself. The change can become observable only as a result of coupling with an additional "external" system which destroys the invariance property in question. If the additional system maintains the invariance property, on the other hand, the change will remain unobservable.

It is precisely through a study of coupling with additional systems that one is led to a definition of the Poisson bracket which is valid under the most general circumstances. The introduction of an additional system is, of course, expressed by a change in the action functional. We shall begin by considering the simplest possible change:

$$S \rightarrow S + \epsilon A \quad , \tag{9.2.7}$$

where A is an invariant of the original system and where the effect of the original system on the added system is neglected, the pertinent dynamical quantities of the latter being lumped into the "constant" ϵ, which will be regarded as small. The change (9.2.7) will induce a change in the dynamical variables ϕ^i, the precise nature of which depends on the boundary conditions selected. For example, we may adopt advanced boundary conditions in which the dynamical states[6] of the system before and after the change are taken to coincide in the remote future, or retarded boundary conditions in which the dynamical states are taken to coincide in the remote past, or a set of boundary conditions intermediate between these two. It is to be noted that the concepts of "past" and "future" require a hyperbolic character for the dynamical equations, but nothing more. Even if the "metric" which determines this hyperbolic character is itself a dynamical variable these concepts retain their validity.

The changes in the ϕ^i corresponding to advanced and retarded boundary conditions will be denoted by $\delta_A{}^+\phi^i$ and $\delta_A{}^-\phi^i$ respectively. The subscript A will sometimes be omitted where no ambiguity can arise, but for the present we keep it. The changes $\delta_A{}^\pm\phi^i$ are, of course, not uniquely determined if the system possesses an invariance group, but are defined only modulo an unobservable transformation (9.2.2). The changes

$$\delta_A{}^{\pm}B \equiv \int B_{,i}\, \delta_A{}^{\pm} \phi^i \, d^4x + O(\epsilon^2) \quad , \tag{9.2.8}$$

in any group invariant B, however, are well defined in virtue of the invariance condition (9.2.5). It will be convenient to introduce the quantity

$$D_A B \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} \delta_A{}^{-}B \quad . \tag{9.2.9}$$

The choice of the retarded boundary condition here, rather than the advanced, anticipates the "one way" character of the measurement process in the description of which this quantity will presently be used.

In the limit of very small ϵ the $\delta_A{}^{\pm}\phi^i$ satisfy the equation

$$\int S_{,ij'}\, \delta_A{}^{\pm} \phi^{j'} \, d^4x' = -\epsilon A_{,i} \quad , \tag{9.2.10}$$

in which the quantities $S_{,ij'}$ and $A_{,i}$ are evaluated using the original values of the dynamical variables. With the "inhomogeneous term" on the right hand side omitted, (9.2.10) becomes the equation which describes the propagation of small disturbances in the system. From its linearity, which permits the application of the superposition principle, the following identities, involving group invariants A, B, C, may be readily inferred:

$$D_A(B + C) = D_A B + D_A C \quad , \tag{9.2.11}$$

$$D_{(A+B)} C = D_A C + D_B C \quad , \tag{9.2.12}$$

$$D_{AB} C = B D_A C + A D_B C \quad . \tag{9.2.13}$$

Furthermore, if c is a numerical coefficient or a variable referring to another system not dynamically coupled to S, then

$$D_{cA} B = c D_A B \quad . \tag{9.2.14}$$

The Poisson bracket of two invariants A and B will, for <u>all</u> physical systems, be defined by

$$(A, B) \equiv D_A B - D_B A \quad . \tag{9.2.15}$$

In the case of systems possessing no dynamical constraints this definition has been shown by Peierls (1952) to reduce to the conventional one. The extension of the definition to the general case will here be made by appealing to the theory of measurement. It will be noted immediately that the usual identities

$$(A, B) = -(B, A) \quad , \tag{9.2.16}$$

$$(A, B+C) = (A, B) + (A, C) \quad , \tag{9.2.17}$$

$$(A, BC) = (A, B)C + B(A, C) \quad , \tag{9.2.18}$$

are satisfied. The verification of the Poisson-Jacobi identity, however, requires an examination of the laws of propagation of disturbances, and will be postponed to the next section.

The system S is formally quantized by relating the commutator to the Poisson bracket in the familiar manner

$$[A, B] = i(A, B) \quad , \tag{9.2.19}$$

which leads immediately to the Uncertainty Principle

$$\Delta A \, \Delta B \gtrsim |\langle (A, B) \rangle| \quad , \tag{9.2.20}$$

where ΔA and ΔB are, for example, the root mean square deviations of A and B respectively from their average values $\langle A \rangle$ and $\langle B \rangle$ in the quantum state in question, and where "$\sim$" means "no smaller in order of magnitude than." We shall devote the remainder of this section to showing that if the Uncertainty Principle holds for one system in the form (9.2.20), with the Poisson bracket given by (9.2.15), then it must hold in that form for <u>all</u> systems. If a description of Nature is demanded which avoids the use of "hidden variables," therefore, the commutator must in all cases be given by Eqs. (9.2.19) and (9.2.15).

The Uncertainty Principle is a statement about the fundamental limitations imposed by the quantum theory on the relation between measurements and the possibilities of making predictions expressed in classical language. Suppose the observable A has been measured with an accuracy ΔA; what does this imply in the way of restrictions on the accuracy of predictions concerning the outcome of subsequent measurements? Before giving a complete answer to this question let us first take note of the fact that the measurement of a given observable will, in general, occupy a finite interval of time, which may itself be involved in the definition of the observable, although in many simple cases this interval may be effectively regarded as vanishingly small. Let us consider the case in which the interval associated with the observable B is subsequent to that associated with A. It will be shown that as a result of the uncontrollable disturbance in the system produced by the measurement of A, the use of a classical value for B in making predictions about the outcome of subsequent measurements of quantities which depend on B is limited to the extent of an uncertainty ΔB which is given by Eq. (9.2.20).

The "classical value" to be used for B in this case is its average value $\langle B \rangle$ in the quantum state resulting from (or "prepared" by) the measurement of A. For simplicity in the subsequent discussion, however, the brackets $< >$ will be omitted whenever it is clear from the context that the "classical value" is meant.

The relation between A and B is a completely reciprocal one, and because of the time reversibility of quantum mechanics the above situation may equally well be described in terms of a limitation ΔA on <u>retrodictions</u> conditioned by a measurement of B with accuracy ΔB in the future. This reciprocity is revealed with particular keenness in the case in which the time intervals associated with A and B overlap. The simplicity of the previous description, in which the measurements of A and B could be ordered in temporal sequence, is missing in this case, and the state of the system must here be regarded as conditioned simultaneously by the results of both measurements, together with their mutual interference. It is the remarkable property of the quantum theory that its formalism consistently mirrors these various cases in such a simple and beautiful way. Furthermore, the generality of this correspondence between formalism and Nature is in complete harmony with the principle of relativity.

Measurements are performed on a system S through coupling with a second system S_a, usually called the <u>apparatus</u>. In principle, any group invariant can be measured through suitable choice of apparatus and coupling. We shall assume that the Uncertainty Principle (9.2.20) holds for the apparatus. It will then follow that it holds also for the system S and, by judicious choice of apparatuses and couplings (those, in fact, by which all physical discoveries have to date been made), may therefore be extended to all (known) physical systems.

We shall begin by considering the measurement of a single observable A. The coupling which is suitable for this measurement is one which brings about a change in the action functional for the combined system of the form

$$S + S_a \rightarrow S + gxA + S_a \quad . \qquad (9.2.21)$$

Here g is an adjustable "coupling constant" and x is some "convenient" apparatus variable. For example, in the Stern-Gerlach experiment, where A is an atomic spin, x may be taken as a finite time integral of the z-component of the position of the atom in a magnetic field which is inhomogeneous in the z-direction, the strength of the field and the magnitude of the atomic magnetic moment being described by g. Likewise, in a field measurement, where A is an average of the field over some space-time domain, x may be a similar time integral of the position of the center of an appropriate test body, the "charge" on the test body being contained in g. The only abstract difference between these two examples is the fact that the eigenvalues of the observable in question come in one case from a discrete set and in the other case from a continuum. Since we are, in this chapter, mainly interested in the latter case we confine our attention to it.

The measurement of A is carried out by determining the deviation in the value of some other suitable apparatus variable π, as a result of the coupling, from the value it would have had in the absence of the coupling. The suitability of the variable π is conditioned by the requirements

$$D_\pi x = 0 \quad , \qquad (x, \pi) = D_x \pi \neq 0 \quad . \qquad (9.2.22)$$

That is, π describes a dynamical state of affairs subsequent to the coupling process, so that although x has a retarded effect on π, π has no retarded effect on x. For example, in the Stern-Gerlach experiment π might be the position of the point at which the atom strikes a photographic plate after having passed through the magnetic field, while in a field measurement π may be the momentum of the test body at the end of the time interval involved in the coupling term gxA, as observed via the Doppler shift of photons, for example. Of course, it is in the analysis of the final observation, performed upon the apparatus variable π, that the source of many of the polemics concerning the conceptual foundations of the quantum theory lies. But the resolution of the difficulties inherent in this analysis, whether in terms of a discontinuous "collapsible" behavior of wave functions, as demanded by the Copenhagen school (see Heisenberg, 1955), or by insistence on an isomorphism between the real world and an infinitely "branching" universal wave function (see Everett, 1957, and Wheeler, 1957), or with the aid of some other viewpoint, is largely a metaphysical problem, irrelevant to the present discussion.

The analysis of the measurement of A itself reduces to simplest form when the coupling term gxA can be regarded as small in comparison with that portion of the apparatus action S_a which corresponds to the time interval involved in gxA.[7] By choosing a sufficiently macroscopic (i.e., "classical") apparatus this can always be arranged. The change in the apparatus variable π as a result of the coupling may then be approximated by

$$\delta^{-}\pi = gAD_x\pi \quad , \qquad (9.2.23)$$

where the factor $D_x\pi$ may be evaluated as if the apparatus were uncoupled to the system S. Actually, Eq. (9.2.23) is not yet sufficiently accurate for our purposes. For, in order to analyze the measurement process in the truly quantum domain it is necessary to take into account small deviations in A from the "classical value" which appears in (9.2.23), in particular, the deviation which is due to the measurement process itself. The latter is given to lowest order by

$$\delta^{-}A = gxD_A A \quad , \qquad (9.2.24)$$

and this is then to be inserted into the improved formula

$$\delta^{-}\pi = g(A + \delta^{-}A)D_x\pi \quad , \qquad (9.2.25)$$

which gives the deviation in π now correct to second order. Here again, the factor $D_A A$ is to be evaluated in the absence of coupling.

Solving Eq. (9.2.25) we obtain the formula

$$A = \frac{\delta^{-}\pi}{g\,D_x\pi} - gx\,D_A A \quad , \qquad (9.2.26)$$

which expresses A in terms of the "experimental data," and from which it follows that the accuracy in the measurement of A will be given by

$$\Delta A \simeq \frac{\Delta\pi}{g|D_x\pi|} + g|D_A A|\Delta x \quad , \qquad (9.2.27)$$

where $\Delta\pi$ and Δx are the uncertainties in the values of π and x in the original apparatus state. Since the Uncertainty Principle is assumed to hold for the apparatus, we have

$$\Delta x\,\Delta\pi \sim |(x,\pi)| = |D_x \pi| \quad , \tag{9.2.28}$$

whence

$$\Delta A \qquad \frac{1}{g\Delta x} + g|D_A A|\Delta x \quad , \tag{9.2.29}$$

which, upon minimization with respect to Δx, reduces to

$$\Delta A \qquad |D_A A|^{\frac{1}{2}} \quad . \tag{9.2.30}$$

In those cases in which the time interval associated with A may be taken vanishingly small (e.g., impulsive measurements in nonrelativistic particle dynamics) the quantity $D_A A$ will usually be equal to zero, and the observable A is then measurable with unlimited accuracy. Unlimited accuracy, however, should be attainable in the measurement of <u>any</u> single observable, and this seems to be contradicted by Eq. (9.2.30) which at first sight implies that there is an absolute limit to the accuracy with which observables associated with finite time intervals can be measured. The way out of this difficulty was found by Bohr and Rosenfeld (1933), who showed that the measuring arrangement can be slightly altered, by means of a "compensation mechanism," in such a way that the correct result is obtained. The compensation mechanism appropriate for the measurement of A is represented by the addition of a term $-\frac{1}{2}g^2 x^2 D_A A$ to the coupling, so that the change in the action now becomes

$$S + S_a \rightarrow S + gxA - \frac{1}{2}g^2 x^2 D_A A + S_a \quad . \tag{9.2.31}$$

Equation (9.2.24) again holds to first order in g . But Eq. (9.2.25) takes the form

$$\delta^- \pi = g(A + \delta^- A - gxD_A A)D_x \pi = gAD_x \pi \quad , \tag{9.2.32}$$

correct to second order in g , whence

$$A = \frac{\delta^- \pi}{gD_x \pi} \quad , \qquad \Delta A \sim \frac{\Delta \pi}{g|D_x \pi|_o} \quad . \tag{9.2.33}$$

Now, the value of $D_A A$ (or, more properly, of $\langle D_A A \rangle$) is generally insensitive to the quantum fluctuations of the system S , being primarily determined by the geometry and parameters of the measuring arrangement and only secondarily, if at all, by the "classical" or "average" values of the system observables. Therefore a compensation device which is adequate for testing the predictions of the quantum theory (e.g., in the case of field measurements, a set of mechanical springs connecting the test body to a stiff coordinate framework) can be set up in advance of the measurement of A on the basis of only a rough prior knowledge of the system observables. The accurate determination of A may therefore be made on the basis of Eq. (9.2.33) with a precision which is limited only by the accuracy with which π may be determined. By choosing the apparatus sufficiently macroscopic the latter accuracy may be made very high indeed without, at the same time, rendering Δx unduly large.

The analysis of the measurement of <u>two</u> observables, A and B , proceeds in a quite similar fashion. Here it is necessary to introduce variables x_1, π_1 and x_2 , π_2 from each of two independent apparatuses, S_{a_1} and S_{a_2} , satisfying the conditions (9.2.22). The systems S_{a_1}

and S_{a_2} may be regarded as forming, together, a single apparatus, for which the Uncertainty Principle will again be assumed to hold. As before, compensation mechanisms will be introduced, but in this case the mutual interference of the two measurements will prevent the complete cancellation of uncertainties. The greatest possible mutual accuracy is attained by means of couplings which produce a change in the total action of the form

$$S + S_{a_1} + S_{a_2} \rightarrow$$

$$S + g_1 x_1 A + g_2 x_2 B - \frac{1}{2} g_1{}^2 x_1{}^2 D_A A - \frac{1}{2} g_1 g_2 x_1 x_2 (D_A B + D_B A)$$

$$- \frac{1}{2} g_2{}^2 x_2{}^2 D_B B + S_{a_1} + S_{a_2} \quad . \tag{9.2.34}$$

The terms in $x_1{}^2$ and $x_2{}^2$ are compensation terms, while the term in $x_1 x_2$ is a <u>correlation term</u> (e.g., in the case of field measurements, resulting from the effect of appropriate mechanical springs connecting the two test bodies involved). As a result of the couplings we have, to first order,

$$\delta^- A = g_1 x_1 D_A A + g_2 x_2 D_B A \quad , \tag{9.2.35}$$

$$\delta^- B = g_1 x_1 D_A B + g_2 x_2 D_B B \quad , \tag{9.2.36}$$

and, to second order,

$$\delta^- \pi_1 = g_1 [A + \delta^- A - g_1 x_1 D_A A - \frac{1}{2} g_2 x_2 (D_A B + D_B A)] D_{x_1} \pi_1$$

$$= g_1 [A - \frac{1}{2} g_2 x_2 (D_A B - D_B A)] D_{x_1} \pi_1 \quad , \tag{9.2.37}$$

$$\delta^- \pi_2 = g_2[B + \delta^- B - \frac{1}{2} g_1 x_1 (D_A B + D_B A) - g_2 x_2 D_B B] D_{x_2} \pi_2$$

$$= g_2[B + \frac{1}{2} g_1 x_1 (D_A B - D_B A)] D_{x_2} \pi_2 \quad , \qquad (9.2.38)$$

whence

$$A = \frac{\delta^- \pi_1}{g_1 D_{x_1} \pi_1} + \frac{1}{2} g_2 x_2 (A,B) \quad , \qquad (9.2.39)$$

$$B = \frac{\delta^- \pi_2}{g_2 D_{x_2} \pi_2} - \frac{1}{2} g_1 x_1 (A,B) \quad , \qquad (9.2.40)$$

leading to the simultaneous accuracy estimates

$$\Delta A \sim \frac{1}{g_1 \Delta x_1} + \frac{1}{2} g_2 |(A,B)| \Delta x_2 \quad , \qquad (9.2.41)$$

$$\Delta B \sim \frac{1}{g_2 \Delta x_2} + \frac{1}{2} g_1 |(A,B)| \Delta x_1 \quad , \qquad (9.2.42)$$

the product of which, upon minimization with respect to the product $\Delta x_1 \Delta x_2$, reduces to

$$\Delta A \Delta B \sim |(A,B)| \quad , \qquad (9.2.43)$$

thus verifying the Uncertainty Principle for the system S .

We may with confidence therefore take the commutator in the form (9.2.19) - (9.2.15) in all future work. It is to be emphasized that the arguments presented here hold with complete generality for all physical systems, including the gravitational field. It is only necessary to make one additional remark, concerning the use of the "classical" or "average" values of the system observables above. Some of these observables may occur in products (in quantities like the slowly varying parts of $D_A B$,

$D_B A$, etc., for example) or may themselves be expressible as products of other observables. Now, the average value of a product may be equated to the product of the average values only in the limit of high quantum numbers, and then only in the case of systems possessing a finite number of degrees of freedom. A rigorous classical description of the quantities in question will therefore not be strictly valid, particularly in the case of quantized fields. Such a description neglects a number of important purely quantum effects, namely, those which give rise to the phenomena of vacuum polarization and level shifts as well as to methematical infinities in the formalism. However, the technical procedure of "renormalization" <u>should</u> reinstate the approximate validity (i.e., to lowest order) of the classical description, <u>provided</u> the coupling of the field to its sources is sufficiently weak and/or there exists a fundamental invariance group which sufficiently dominates the physics. At least this is the case for quantum electrodynamics, as has been emphasized by Bohr and Rosenfeld(1950). The gravitiational field, also, certainly meets these specifications, although in this case the procedures for renormalization are still unknown. One hopes to be able to lump at least some of the infinities together into a renormalization of the gravitation constant, but this remains to be seen. In the following sections we shall refer to the use of the classical description for all quantities occuring in the derivation of a Poisson bracket (except those, of course, which appear in the primary commutator which the Poisson bracket evaluates) as the <u>semi-classical approximation</u>. In the derivations of the semi-classical approximation all quantites are regarded as freely commutable c-numbers. The problem of their actual non-commutability will be only briefly considered at appropriat points in the discussion.

(9.3) Green's functions.

The laws of propagation of small disturbances in the system S are determined by the fundamental structure $S_{,ij'}$ appearing in Eq. (9.2.10). It is convenient to treat this structure formally as a continuous matrix although typically it, like $R^i{}_{L'}$, will actually be a differential operator, expressible in matrix form as a linear combination of the delta function and its derivatives up to some finite order (usually first or second) with coefficients involving the ϕ^i and their derivatives up to some finite order. Because variational differentiation is commutable $S_{,ij'}$ is a symmetric matrix. It is also a singular matrix whenever the system possesses an infinite dimensional invariance group. This follows from Eq. (9.2.6), which admits the corollary

$$\int S_{,ij'} R^{j'}{}_A \, d^4x' = 0 \tag{9.3.1}$$

whenever the dynamical equations are satisfied. The $R^i{}_A$, because of their "locality" (i.e., they vanish except in the immediate neighborhood of z, for each z), are true null eigenvectors.

Because of the singularity of $S_{,ij'}$ the solutions $\delta_A{}^{\pm}\phi^i$ of Eq. (9.2.10) (as has already been pointed out) are not well defined but are determined only up to a group transformation (9.2.2). It is evident that the general solution of Eq. (9.2.10) is obtained by adding (9.2.2) to an arbitrary linear combination of particular solutions (with coefficients adding up to unity) determined by appropriate boundary and supplementary conditions. The boundary conditions to be adopted are already implied by the $\pm$ signs. For the supplementary condition it is necessary to choose an equation of the form

$$\int Q_{iA}\, \delta_A{}^{\pm}\, \phi^i\, d^4x = 0 \quad , \tag{9.3.2}$$

where Q_{iA} , like $R^i{}_A$, is a differential operator which may be dependent on the ϕ^i , but which is selected in such a way that still another similar differential operator $P_i{}^A$ may be found for which the matrices

$$F_{ij'} \equiv S_{,ij'} + \int P_i{}^A\, Q_{j'A}\, d^4z \quad , \tag{9.3.3}$$

$$F_{AB'} \equiv \int Q_{iA}\, R^i{}_{B'}\, d^4x \quad , \tag{9.3.4}$$

$$F_A{}^{B'} \equiv \int R^i{}_A\, P_i{}^{B'}\, d^4x \quad , \tag{9.3.5}$$

are all nonsingular. In the theory of discrete matrices "vectors" $P_i{}^A$, Q_{iA} , $R^i{}_A$ having these properties are easily found by identifying pertinent subspaces, and considerable flexibility is allowed in their selection. The same is true of these quantities in the case of all action functionals which lead to consistent dynamical theories. Furthermore, because of the underlying hyperbolic character of the dynamical equations of these theories, the matrices $F_{ij'}$, $F_{AB'}$, $F_A{}^{B'}$ may be chosen so as to possess special properties which allow us to characterize them as <u>wave operators</u>.

A wave operator (let us refer to $F_{ij'}$ for definiteness) satisfies the following two conditions: (1) it admits of bounded nonvanishing solutions $\delta\phi^i$ to the equation

$$\int F_{ij'}\,\delta\phi^{j'}\,d^4x' = 0 \quad ; \tag{9.3.6}$$

and (2) it possesses unique retarded and advanced Green's functions $G^{\pm ij'}$ satisfying the equations

$$\int F_{ik''}\,G^{\pm k''j'}\,d^4x'' = -\,\delta_i^{\ j'} \quad , \tag{9.3.7a}$$

$$\int G^{\pm ik''}\,F_{k''j'}\,d^4x'' = -\,\delta_{j'}^{\ i} \quad , \tag{9.3.7b}$$

and the conditions

$$\left.\begin{aligned} G^{-ij'} &= 0 \quad \text{for} \quad x < x' \quad , \\ G^{+ij'} &= 0 \quad \text{for} \quad x > x' \quad . \end{aligned}\right\} \tag{9.3.8}$$

Here the symbol $\delta_i^{\ j'}$ denotes in obvious fashion a product of a Kronecker delta with a delta function, while "<" is an abbreviation for "lies to the past of" and ">" is an abbreviation for "lies to the future of." In a space-time with hyperbolic metric the definitions of "past" and "future" may be made with respect to an arbitrary space-like hypersurface through either one or the other of the two points x , x' . Because of the arbitrariness of this hypersurface it follows that both Green's functions vanish simultaneously when x and x' are separated by a space-like geodetic interval.

It will be seen presently that Eqs. (9.3.7a) and (9.3.7b) are not independent; one follows from the other just as in the case of finite matrices. It is only necessary to bear in mind that the use of the latter equation entails an integration by parts, the admissibility of which must be checked in context.[8] It is to be noted, however, that $F_{ij'}$, unlike a nonsingular finite matrix, does not possess a unique inverse; both $-G^{-ij'}$ and $-G^{+ij'}$, as well as linear combinations of the two, are its "inverses." This fact is a direct consequence of the existence of bounded solutions to Eq. (9.3.6), which may always be added to any "inverse." On the other hand, it is to be recognized that bounded solutions of Eq. (9.3.6) cannot vanish in remote regions of space-time sufficiently rapidly to be normalizable (i.e., quadratically integrable). For if they did, then $F_{ij'}$ would possess true null eigenvectors and have no inverses at all.

Consider, now, two arbitrary functions $\Phi_1{}^i$, $\Phi_2{}^i$, which appear together with the wave operator $F_{ij'}$ in the following combination:

$$\int(\Phi_1{}^i\, F_{ij'}\, \Phi_2{}^{j'} - \Phi_1{}^{j'}\, F_{j'i}\, \Phi_2{}^i) d^4x' \quad .$$

If the functions $\Phi_1{}^i$, $\Phi_2{}^i$ vanish sufficiently rapidly in remote regions of space-time, the integral of this expression over all x will vanish by symmetry. Since $F_{ij'}$ is a differential operator this implies that the above integral must be reexpressible in the form

$$\int(\Phi_1{}^i F_{ij'}\Phi_2{}^{j'} - \Phi_1{}^{j'}F_{j'i}\Phi_2{}^i)d^4x' = \int d^4x' \int d^4x'' \frac{\partial}{\partial x^\mu}(\Phi_1{}^{i'} f^\mu{}_{i'j''}\Phi_2{}^{j''}) \quad , \tag{9.3.9}$$

where $f^\mu{}_{i'j''}$ is an appropriate homogeneous quadratic combination of delta functions and their derivatives, with coefficients involving the ϕ^i and their derivatives. Since the identity (9.3.9) involves the properties of the functions $\Phi_1{}^i$, $\Phi_2{}^i$ only locally, it must evidently hold for arbitrary functions $\Phi_1{}^i$, $\Phi_2{}^i$, subject only to the conditions which permit integrations by parts.

With the aid of this identity we may show that the Green's functions $G^{\pm ij'}$ have the very important property of being able, in the combination

$$G^{ij'} \equiv G^{+ij'} - G^{-ij'} \quad , \tag{9.3.10}$$

to express Huygens' principle for a solution $\delta\phi^i$ of Eq. (9.3.6):

$$\delta\phi^i = \int_\Sigma d\,\Sigma_{\mu'} \int d^4x'' \int d^4z\, G^{ij''} f^{\mu'}{}_{j''a}\, \delta\phi^a \quad . \tag{9.3.11}$$

Here the value at an arbitrary point x of the solution $\delta\phi^i$ is expressed in terms of Cauchy data,

$$\int f^{\mu'}{}_{j''a}\, \delta\phi^a\, d^4z \quad , \tag{9.3.12}$$

on a space-like hypersurface Σ having directed surface element $d\Sigma_{\mu'}$. The proof of Eq. (9.3.11) is carried out by changing the surface integral into a volume integral with the aid of Gauss' theorem, and then using Eq. (9.3.9). For $x > \Sigma$ Eq. (9.3.11) becomes

$$\delta\phi^i = \int_\Sigma^{\text{future}} d^4x' \int d^4x'' (G^{-ij'} F_{j'k''} \delta\phi^{k''} - G^{-ik''} F_{k''j'} \delta\phi^{j'}) \quad , \tag{9.3.13}$$

while for $x < \Sigma$ it becomes

$$\delta\phi^i = \int_{\text{past}}^\Sigma d^4x' \int d^4x'' (G^{+ij'} F_{j'k''} \delta\phi^{k''} - G^{+ik''} F_{k''j'} \delta\phi^{j'}) \quad , \tag{9.3.14}$$

the validity of both forms following immediately from Eqs. (9.3.6) and (9.3.7b). The extension of the domains of integration arbitrarily far into the future and past respectively is permitted because of the "locality" of $F_{ij'}$ and the fact that the Green's functions in each case "cut-off" sharply beyond the point x. In the case of x lying on Σ Eq. (9.3.11) is to be regarded as providing an interpretation of the singularities of the $G^{\pm ij'}$ and their derivatives, regarded as functions of x', in the space-like neighborhood of x.

It will be noted that Eq. (9.3.7b) was used in the above derivation, but not Eq. (9.3.7a). If we therefore take Eq. (9.3.7b) as the defining equation for the Green's functions we may infer the validity of Eq. (9.3.7a) through the following considerations: Because the functions $\delta\phi^i$ satisfy Eq. (9.3.6) and because the Cauchy data (9.3.12) may be chosen completely arbitrarily on Σ, it follows from Eq. (9.3.11) that the function $G^{ij'}$, which is known as the propagation function for the wave operator $F_{ij'}$, also satisfies Eq. (9.3.6), i.e.,

$$\int F_{ik''} G^{k''j'} d^4x'' = 0 \quad , \qquad (9.3.15a)$$

as well as the equation

$$\int G^{ik''} F_{k''j'} d^4x'' = 0 \quad , \qquad (9.3.15b)$$

which follows immediately from Eqs. (9.3.7b) and (9.3.10). Equation (9.3.7a) is then obtained by splitting the propagation function appearing in Eq. (9.3.15a) into its advanced and retarded parts. The kinematics of these parts insure that it is only the delta function $\delta_i{}^{j'}$ or its derivatives which can make an appearance on the right hand side, while dimensional considerations eliminate the latter. The coefficient of the delta function is determined as -1 from the identity

$$\int d^4z \int d^4z' \, G^{\pm ia} F_{ab'} G^{\pm b'j'} = -G^{\pm ij'} \quad , \qquad (9.3.16)$$

in which integration by parts and interchange of orders of integration is permitted in virtue of the conditions (9.3.8).

Returning now to Eq. (9.2.10), we see that it may be replaced by

$$\int F_{ij'} \delta_A^{\pm} \phi^{j'} d^4x' = -\epsilon A_{,i} \quad , \qquad (9.3.17)$$

from which one immediately obtains

$$\delta_A^{\pm} \phi^i = \epsilon \int G^{\pm ij'} A_{,j'} d^4x' \quad , \qquad (9.3.18)$$

whenever the supplementary condition (9.2.10) is satisfied. If the supplementary condition is not already satisfied by the $\delta_A^{\pm} \phi^i$ it is easily imposed by first carrying out a group transformation (9.2.2) for which the parameters $\delta\xi^A$ are given by

$$\delta\xi^{\pm} = \int d^4x \int d^4z' \, G^{\pm AB'} Q_{iB'} \delta_A^{\pm} \phi^i \quad , \qquad (9.3.19)$$

the $G^{\pm AB'}$ being the Green's functions for the wave operator $F_{AB'}$. It is important to check, however, that the solutions (9.3.18) in fact satisfy the supplementary condition which was used to get them in the first place. This can be done with the aid of an important relation between the Green's functions $G^{\pm ij'}$ and those belonging to the wave operator $F_A{}^{B'}$. We note, using Eqs. (9.3.1), (9.3.3), (9.3.5) and the symmetry of $S_{,ij'}$, that

$$\int R^{j'}{}_A F_{j'i} d^4x' = \int F_A{}^{B'} Q_{iB'} d^4z' \quad . \qquad (9.3.20)$$

Therefore

$$\int d^4x \int d^4z' \, F_A{}^{B'} Q_{iB'} \, G^{\pm ij'} = \int d^4x \int d^4x' \, R^{k''}{}_A \, F_{k''i} \, G^{\pm ij'} = -R^{j'}{}_A \, . \tag{9.3.21}$$

But also

$$\int d^4z' \int d^4z'' \, F_A{}^{B'} \, G^{\pm}{}_{B'}{}^{C''} \, R^{j'}{}_{C''} = -R^{j'}{}_A \, , \tag{9.3.22}$$

where the $G^{\pm}{}_{B'}{}^{C''}$ are the Green's functions for $F_A{}^{B'}$. Now, Eqs. (9.3.21) and (9.3.22) are both "wave equations" in $F_A{}^{B'}$, having the same inhomogeneous term, $-R^{j'}{}_A$. The functions satisfying these equations have the same kinematical properties and must therefore be identical. That is,

$$\int Q_{iA} \, G^{\pm ij'} \, d^4x = \int G^{\pm}{}_A{}^{B'} \, R^{j'}{}_{B'} \, d^4z' \quad , \tag{9.3.23}$$

which is the relation mentioned. Using it we get from Eq. (9.3.18) immediately

$$\int Q_{iA} \, \delta_A{}^{\pm} \varphi^i \, d^4x = \epsilon \int d^4x' \int d^4z' \, G^{\pm}{}_A{}^{B'} \, R^{j'}{}_{B'} \, A_{,j'} \, , \tag{9.3.24}$$

which vanishes in virtue of the group invariance of A, thus showing the complete self-consistency of the supplementary condition.

It is convenient at this point to derive also another relation similar to (9.3.23). Using Eqs. (9.3.1), (9.3.3) and (9.3.4), we have

$$\int F_{ij'} \, R^{j'}{}_A \, d^4x' = \int P_i{}^{B'} \, F_{B'A} \, d^4z' \quad , \tag{9.3.25}$$

and hence

$$\int d^4x' \int d^4z' \, G^{\pm ij'} P_{j'}{}^{B'} F_{B'A} = \int d^4x' \int d^4x'' \, G^{\pm ij'} F_{j'k''} R^{k''}{}_A = - R^i{}_A \, . \tag{9.3.26}$$

But also

$$\int d^4z' \int d^4z'' \, R^i{}_{C''} G^{\pm C''B'} F_{B'A} = - R^i{}_A \, , \tag{9.3.27}$$

from which it may be inferred that

$$\int G^{\pm ij'} P_{j'}{}^{B'} d^4x' = \int R^i{}_A G^{\pm AB'} d^4z \, . \tag{9.3.28}$$

This relation is useful in the derivation of an important reciprocity theorem involving the Green's functions $G^{\pm ij'}$. We first write

$$\int F_{ik''} (G^{\pm k''j'} - G^{\mp j'k''}) d^4x''$$

$$= - \int (F_{ik''} - F_{k''i}) G^{\mp j'k''} d^4x''$$

$$= - \int d^4x'' \int d^4z \, (P_i{}^A Q_{k''A} - P_{k''}{}^A Q_{iA}) G^{\mp j'k''} \, . \tag{9.3.29}$$

The solution of this equation, taking into account the kinematics of the Green's functions and using (9.3.28), is

$$G^{\pm ij'} - G^{\mp j'i}$$

$$= \int d^4x'' \int d^4z \int d^4z' (R^i{}_{B'} G^{\pm B'A} G^{\mp j'k''} - R^{j'}{}_{B'} G^{\mp B'A} G^{\pm ik''}) Q_{k''A} \, . \tag{9.3.30}$$

From this it follows, with the aid of Eq. (9.3.18) and the invariance condition (9.2.5), that if A and B are any two group invariants, then

$$\delta_A^{\pm}B - \delta_B^{\mp}A = \int (B_{,i}\, \delta_A^{\pm} \phi^i - A_{,i}\, \delta_B^{\mp} \phi^i) d^4x$$

$$= \epsilon \int d^4x \int d^4x'\, B_{,i} (G^{\pm ij'} - G^{\mp j'i}) A_{,j'}$$

$$= 0 \quad . \tag{9.3.31}$$

That is, the retarded effect of A on B is equal to the advanced effect of B on A, and vice versa.

This reciprocity theorem allows us to write the Poisson bracket of A and B in the following simple form:

$$(A,B) = D_A B - D_B A = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (\delta_A^- B - \delta_B^- A)$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} (\delta_B^+ A - \delta_B^- A)$$

$$= \int d^4x \int d^4x'\, A_{,i}\, G^{ij'}\, B_{,j'} \quad , \tag{9.3.32}$$

in which the propagation function appears. It is to be emphasized that the value of the final expression is independent of variations in the propagation function arising from varying choices of the somewhat arbitrary functions $P_i^{\ A}$, Q_{iA}, $R^i_{\ A}$. This, of course, is obvious from the original definition of the Poisson bracket, but it can also be proved directly by studying the transformation properties of the various Green's functions under allowable transformations of the $P_i^{\ A}$, Q_{iA}, $R^i_{\ A}$, and making use of the invariance properties of A and B. It may be mentioned that the functions $P_i^{\ A}$ and Q_{iA} can in practice usually be chosen in such a way that the matrix $F_{ij'}$ is symmetric (self-adjoint wave operator). In this circumstance

$$G^{\pm ij'} = G^{\mp j'i} \quad , \tag{9.3.33}$$

and the Poisson bracket identities (9.2.16), (9.2.17), (9.2.18) may be read off immediately from the final expression in (9.3.32).

A more important consequence of the reciprocity theorem is its significance for the theory of the <u>canonical transformation group</u>. The existence of such a group is shown by the following considerations: Since, in the limit of infinitesimal ϵ ,

$$\int S_{,ij'}(\delta_B^{+}\phi^{j'} - \delta_B^{-}\phi^{j'})d^4x' = 0 \tag{9.3.34}$$

it follows that the quantities $\phi^i + \delta_B^{+}\phi^i - \delta_B^{-}\phi^i$ satisfy the dynamical equations if the ϕ^i do. By means of the Poisson bracket, therefore, invariants may be used to map solutions of the dynamical equations into other solutions. For example, the invariant B defines the infinitesimal mapping

$$A \rightarrow T(B)A = A + \epsilon(A,B) \quad , \quad \text{for all } A \ , \tag{9.3.35}$$

where, in virtue of the reciprocity theorem, the symbol $T(B)$ may be expressed in the form

$$T(B) \equiv 1 + \delta_B \quad , \tag{9.3.36}$$

$$\delta_B \equiv \delta_B^{+} - \delta_B^{-} \quad , \qquad \delta_{BA} = \epsilon(A,B) \quad , \tag{9.3.37}$$

the symbols $\delta_B^{\pm}$ being viewed here in their evident role as linear operators. The mapping (9.3.35) is expressed in terms of its effect on the class of all invariants, since it is only in terms of invariants that physically distinct solutions of the dynamical equations may be characterized. It is easy to see that such mappings are one-to-one, at least in the neighborhood of the identity, and therefore generate a group.[9] From this fact it follows that the Poisson bracket (9.2.15) satisfies not only the identities (9.2.16), (9.2.17), (9.2.18), but also the Poisson-Jacobi identity as well; for the Poisson brackets may be mapped into the commutators of the <u>Lie ring</u> associated with the mapping group.

To see how this comes about we first note that the result, $T(A)X$, of an infinitesimal mapping $T(A)$ performed on an arbitrary invariant X may be regarded in either of two guises: (1) as an invariant which differs slightly from X, or (2) as the same invariant, but evaluated with a set of dynamical variables differing slightly --- but physically --- from the original variables. Therefore, if we consider the product of two successive infinitesimal mappings, $T(A)$ and $T(B)$, we may write, using the first point of view, simply

$$T(B)\,T(A)X = (1 + \delta_A + \delta_B + \delta_B\delta_A)X \quad . \qquad (9.3.38)$$

Using the second point of view, however, and making the functional dependence of the invariants A, B, X on the dynamical variables ϕ^i explicit, we may write

$$\begin{aligned}
&T(B[\phi])\,T(A[\phi])\,X[\phi] \\
&\qquad = T(B[\phi + \delta_A\phi] - \delta_A B[\phi])\,X[\phi + \delta_A\phi] \\
&\qquad = (1 + \delta_{B[\phi + \delta_A\phi]} - \delta_{\delta_A B[\phi]})\,X[\phi + \delta_A\phi] \\
&\qquad = T(A)\,T(B)\,X - \delta_{\delta_A B}\,X \\
&\qquad = (1 + \delta_A + \delta_B + \delta_A\delta_B - \delta_{\delta_A B})X \quad . \qquad (9.3.39)
\end{aligned}$$

Equating the right hand sides of Eqs. (9.3.38) and (9.3.39), we infer, from the arbitrariness of X, therefore,

$$[\delta_A , \delta_B] = \delta_{\delta_A B} = -\delta_{(A,B)} , \qquad (9.3.40)$$

and hence

$$0 \equiv [\delta_A , [\delta_B, \delta_C]] + [\delta_B, [\delta_C, \delta_A]] + [\delta_C, [\delta_A, \delta_B]]$$

$$= \delta_{\epsilon^2[(A,(B,C)) + (B,(C,A)) + (C,(A,B))]} , \qquad (9.3.41)$$

which, in virtue of the fact that $\delta_X = 0$ if and only if $X = 0$, implies

$$(A,(B, C)) + (B,(C, A)) + (C,(A, B)) = 0 . \qquad (9.3.42)$$

This identity may also be proved using Eq. (9.3.32), by working directly with the Green's functions (DeWitt, 1961).[10]

In evaluating Poisson brackets by means of Eq. (9.3.32) a possible source of ambiguity at first sight appears to exist. Heretofore, in referring to group invariants, we have always had in mind explicit functional expressions involving the ϕ^i. Actually, invariants are defined only modulo the dynamical equations. It is straightforward to show, however, that this freedom leaves the value of the Poisson bracket unaffected. Let us, for example, replace B by

$$B' = B + \int f^i S_{,i} \, d^4x , \qquad (9.3.43)$$

where the f^i are arbitrary coefficients. [The group invariance of the second term follows from Eq. (9.3.1) together with the dynamical equations.] We have

$$(A, B') = (A, B) + \int d^4x \int d^4x' \int d^4x'' \, A_{,i} \, G^{ij'} \, S_{,j'k''} \, f^{k''} \quad , \tag{9.3.44}$$

in which terms in $S_{,i}$ have been dropped <u>after</u> the variational differentiations have been performed. In virtue of Eqs. (9.3.3), (9.3.15b) and (9.3.28), however, this becomes

$$(A, B') = (A, B) - \int d^4x \int d^4x'' \int d^4z \int d^4z' \, A_{,i} \, R^i{}_A \, G^{AB'} Q_{k''B'} \, f^{k''} \quad , \tag{9.3.45}$$

which reduces simply to (A, B) in view of the invariance of A.

All of the preceding work has been carried out in the semi-classical approximation with all quantities being treated as freely commutable c-numbers. We may here briefly indicate some of the problems which arise in the rigorous theory. In the first place, the use of quantities which are commutable in lowest approximation restricts the application of the theory to systems satisfying Bose statistics. In the semi-classical approximation it is actually not difficult to extend the theory to include Fermi systems as well. The details of this extension are outlined in Appendix A at the end of the chapter. It is only necessary to introduce anticommuting as well as commuting "c-numbers." Beyond that, however, the problems become difficult. The quantities in the rigorous theory do not exactly commute or anticommute, and the order of factors must be taken into detailed account. It is no longer clear to what extent the formalism is determined by physics alone and to what extent it is determined by purely mathematical exigencies (not that one expects the two to be separable in the end, of course). The difficulties are increased by the fact that the Green's functions are themselves q-numbers in all except completely trivial linear theories. In the case of systems possessing infinite dimensional invariance groups the way out of these difficulties is completely unknown except when the theory is nearly linear (e.g., quantum electrodynamics). When infinite dimensional invariance groups are absent it is possible to give some indication as to how one may proceed. In this case commutation relations may be written directly for the ϕ^i themselves, namely

$$[\phi^i, \phi^{j'}] = i\, G^{ij'} \quad . \tag{9.3.46}$$

The problem which then arises is that of defining the "operator-propagator" $G^{ij'}$ in such a way that it can really be a commutator. In particular, it must be consistent with the Poisson-Jacobi identity as well as with the dynamical equations. Its consistency with the latter provides a possible clue. Suppose the dynamical equations $S_{,i} = 0$ have been written with their factors in some given order. Then by taking the commutator of the dynamical equations with $\phi^{j'}$, we obtain

$$0 = [S_{,i}, \phi^{j'}] = i \int S_{,ik''} \cdot G^{k''j'}\, d^4x \quad , \tag{9.3.47}$$

where the dot signifies that the propagation function is to be inserted as a replacement for $\delta\phi^{k''}$ in all the places in which it occurs in the variation $\delta S_{,i}$. This then suggests that the Green's functions themselves be defined by

$$\int S_{,ik''} \cdot G^{\pm k''j'}\, d^4x'' = -\,\delta_i^{\;j'} \quad . \tag{9.3.48}$$

It is not obvious, however, what conditions the original structure

$S_{,i}$ has to satisfy in order that Green's functions defined in this way satisfy the necessary reciprocity relations (9.3.33) and, finally, that the Poisson-Jacobi identity hold. It may be possible, nevertheless to build up a self-consistency scheme, by successive approximations perhaps, to answer this question. Incidentally, there is no a priori reason to insist that $S_{,i}$ should be the variational derivative of some actual action operator. Its full expression may require the addition of some small (proportional to $\hbar^2$) "non-classical" terms. Its complete determination may also depend on a number of other considerations, for example, on the requirement that all criteria adopted be invariant under transformations which replace the ϕ^i, as primary dynamical variables, by arbitrary local functions of themselves.

When infinite dimensional invariance groups are present it may be necessary first to introduce some "preferred" invariant dynamical variables (determined in the case of general relativity, for example, by an intrinsic coordinate system based either on the geometry of space-time itself or on some additional physical system) which satisfy a set of invariant dynamical equations and for which a commutator like (9.3.46) can be written. On the other hand, it might prove possible to deal with the original dynamical variables as if they satisfied the commutation relation (9.3.46), as long as all final expressions involve only group invariants. The commutator of two invarients A and B would then take the form

$$[A, B] = i\int d^4x \int d^4x' \, A_{,i} \cdot G^{ij'} \cdot B_{,j'} , \qquad (9.3.49)$$

the condition for invariance itself becoming

$$\int A_{,i} \cdot R^i{}_A \, d^4x = 0. \tag{9.3.50}$$

In Eq. (9.3.49) the pair of dots signify that the propagation function is first to be inserted as a replacement for $\delta\phi^{j'}$ in all the places in which it occurs in the variation δB and that the resulting "product" is then to be inserted as a replacement for $\delta\phi^i$ in the variation δA, or, alternatively, that the process of insertion is first performed in δA and then in δB. The equivalence of the two procedures follows from familiar properties of commutator brackets, together with the assumption that $G^{ij'}$ is itself actually a commutator. Whether complete consistency of the quantum theory of geometry, in particular, can be established along these lines remains to be seen.

We conclude this section by showing how the arguments and methods thus far introduced can be used to derive the generator of infinitesimal displacements in space-time. It is convenient for this purpose to work with an action functional which is formally invariant under the group of general transformations of the coodinates x^μ, even though the system in question may not really be invariant under this group. Such an action functional can always be constructed simply by starting with a standard "simplest" form and performing an arbitrary coordinate transformation. Metric components $g_{\mu\nu}$ will then make an appearance as explicit functions of the x^μ. In this way nonrelativistic and Lorentz invariant theories, as well as generally covariant theories with fixed gravitational fields, can all be treated at once.[11] It will become apparent in Section 7, furthermore, that the method also covers the fully relativistic situation in which the gravitational field itself is given dynamical properties. In this case it is only necessary to use intrinsic coordinates in place of the x^μ.

An infinitesimal displacement δx^μ of the coordinate mesh corresponds to the coordinate transformation $x'^\mu = x^\mu - \delta x^\mu$. Under this transformation the functional form of the action suffers an explicit change of amount

$$\delta S \equiv \frac{1}{2}\int T^{\mu\nu}\, \delta g_{\mu\nu}\, d^4x \quad , \tag{9.3.51}$$

where $T^{\mu\nu}$ is the stress-energy density of the system and $\delta g_{\mu\nu}$ is the change which the mesh displacement induces in the explicit metric $g_{\mu\nu}$:

$$T^{\mu\nu} = 2\, \delta S/\delta g_{\mu\nu} \quad , \tag{9.3.52}$$

$$\delta g_{\mu\nu} \equiv \delta x_{\mu.\nu} + \delta x_{\nu.\mu} \quad . \tag{9.3.53}$$

Here the dot denotes the covariant derivative with respect to the metric $g_{\mu\nu}$. In order to insure convergence of the integral (9.3.51), δx^μ will be required to vanish outside of a finite but otherwise arbitrary region of space-time.

Since the action functional is formally coordinate invariant the change (9.3.51) will be exactly cancelled by a variation in the ϕ^i corresponding to the same coordinate transformation: $x'^\mu = x^\mu - \delta x^\mu$. This fact is of great importance, since it means that the change (9.3.51) can also be computed by taking the negative of the variation induced in S by replacing the ϕ^i and their derivatives by their "displaced" values. It is to be noted, however, that when the dynamical equations are satisfied this latter variation vanishes on account of the stationary action principle. This means that expression (9.3.51) itself has vanishing value, and hence

$$0 = \delta S = \int T^{\mu\nu} \delta x_{\mu\cdot\nu} \, d^4x = -\int T^{\mu\nu}{}_{\cdot\nu} \, \delta x_\mu \, d^4x \quad , \quad (9.3.54)$$

which, in view of the arbitrariness of δx^μ , implies

$$T^{\mu\nu}{}_{\cdot\nu} = 0 \qquad (9.3.55)$$

In spite of the fact that expression (9.3.51) has vanishing value its explicit form is significant, and can be viewed just as if it represented a real change in the physical system. The vanishing of δS simply means that the retarded and advanced effects which it produces will be identical.

Let us consider an arbitrary local tensor quantity constructed out of the ϕ^i and their derivatives, the components of which may be imagined as arranged in a 1-column array . Let us further suppose that Φ is a group invariant of the system. The change in Φ produced by the change δS in the action will vanish in regions where δx^μ vanishes and will elsewhere take the form

$$\delta^+\Phi = \delta^-\Phi = \Phi_{\cdot\mu}\delta x^\mu + D_\mu{}^\nu \Phi \delta x^\mu{}_{\cdot\nu} \quad , \qquad (9.3.56)$$

corresponding to the alteration in the coordinate mesh with respect to which Φ is viewed. Here the $D_\mu{}^\nu$ are generators of the matrix representation of the <u>linear group</u> to which Φ corresponds; they satisfy the commutation relations

$$[D_\mu{}^\sigma , D_\nu{}^\tau] = \delta_\mu^\tau D_\nu{}^\sigma - \delta_\nu{}^\sigma D_\mu^\tau \quad . \qquad (9.3.57)$$

We also have, however,

$$\delta^{\pm}\Phi = \int d^4z \int d^4z' \frac{\delta\Phi}{\delta\phi^a} G^{\pm ab'} \frac{\delta\,\delta S}{\delta\phi^{b'}}$$

$$= \begin{cases} \displaystyle\int d^4x' \int d^4z \int d^4z' \frac{\delta\Phi}{\delta\phi^a}\, \theta(z',z)\, G^{ab'} \frac{\delta T^{\mu'\nu'}}{\delta\phi^{b'}} \delta x_{\mu'.\nu'} \\ \displaystyle -\int d^4x' \int d^4z \int d^4z' \frac{\delta\Phi}{\delta\phi^a}\, \theta(z,z')\, G^{ab'} \frac{\delta T^{\mu'\nu'}}{\delta\phi^{b'}} \delta x_{\mu'.\nu'}\ , \end{cases} \tag{9.3.58}$$

where $\theta(z,z')$ is the <u>step function</u>:

$$\theta(z,z') = \begin{cases} 1 & \text{when } z > z' \\ 0 & \text{when } z < z' \end{cases} \quad . \tag{9.3.59}$$

Because of the "locality" of Φ and $T^{\mu'\nu'}$ the variational derivatives appearing in (9.3.58) will consist of linear combinations of the delta functions $\delta(x,z)$ and $\delta(x',z')$ and their derivatives, x being the point at which Φ is evaluated. This means that it is <u>almost</u> permitted to replace the step functions $\theta(z',z)$, $\theta(z,z')$ by $\theta(x',x)$, $\theta(x,x')$ respectively. In fact this replacement <u>can</u> be made provided extra terms involving the derivatives of the step functions are added as needed in order to account for the effect of the differentiated delta function. Lumping these extra terms collectively into the symbol $\Delta\Phi$, we may therefore write

$$\delta^{\pm}\Phi = \begin{cases} \displaystyle (\Phi, \int_{\Sigma}^{\text{future}} T^{\mu'\nu'}\, \delta x_{\mu'.\nu'}\, d^4x') + \Delta\Phi \\ \displaystyle -(\Phi, \int_{\text{past}}^{\Sigma} T^{\mu'\nu'}\, \delta x_{\mu'.\nu'}\, d^4x') + \Delta\Phi\ , \end{cases} \tag{9.3.60}$$

where Σ is an arbitrary space-like hypersurface through x. Performing an integration by parts and making use of Eq. (9.3.55), we finally obtain

$$\Phi_{\cdot\mu}\,\delta x^{\mu} + D_{\mu}{}^{\nu}\,\Phi\,\delta x^{\mu}{}_{\cdot\cdot\nu} = (\Phi, \mathfrak{H}) + \Delta\Phi \quad , \tag{9.3.61}$$

$$\mathfrak{H} \equiv -\int_{\Sigma} T_{\mu'}{}^{\nu'}\,\delta x^{\mu'}\, d\Sigma_{\nu'} \quad . \qquad (9.\ .62)$$

It is seen that an infinitesimal displacement can be effected on a group invariant Φ by means of a simple Poisson bracket with an infinitesimal generator $\mathfrak{H}$ --- which is the nearest thing to a Hamiltonian appearing in the present formalism --- only if $\Delta\Phi$ vanishes. $\Delta\Phi$ will vanish or not depending on the effect of the singularities possessed by the products of the propagation function with derivatives of the step function. Generally speaking, in a theory for which the dynamical equations are of the second differential order in the ϕ^{i} , $\Delta\Phi$ will vanish if Φ depends only on the ϕ^{i} but not on their derivatives. That $\Delta\Phi$ will not generally vanish when Φ depends on the derivatives of ϕ^{i} is then easily seen by taking derivatives of Eq. (9.3.61) and remembering that $\mathfrak{H}$ itself depends on x through its dependence on Σ. This, however, in turn implies that $\Delta\Phi$ may in special cases vanish for all Φ, namely, if the displacement δx^{μ} can be chosen in such a way that $\mathfrak{H}$ becomes independent of Σ.[12] In such cases we may speak of a "true Hamiltonian" for the system. It is to be emphasized once again, however, that although its use is often a convenience, the infinitesimal displacement generator is not essential to the quantization program.

(9.4) The free particle and the relativistic clock.

In this section we apply the Green's function techniques to two simple examples: the nonrelativistic free particle and the relativistic clock. The first provides a familiar introduction to method while the second is of fundamental importance in the theory of measurement of the geometry of space-time.

The action functional of the free particle may be taken in the standard form

$$S \equiv \frac{1}{2}m \int \dot{\underset{\sim}{x}}^2 \, dt \quad , \qquad (9.4.1)$$

where m is the mass and $\underset{\sim}{x} = (x_1, x_2, x_3)$ the position vector of the particle, and where the dot denotes differentiation with respect to the time t. The variational derivatives

$$\delta S/\delta x_i \equiv - m \ddot{x}_i \ , \qquad (9.4.2)$$

$$\delta^2 S/\delta x_i \, \delta x_{j'} \equiv -m \, \delta_{ij} \, \ddot{\delta}(t-t') \quad , \qquad (9.4.3)$$

lead to the dynamical equations

$$-m\ddot{\underset{\sim}{x}} = 0 \qquad (9.4.4)$$

and to the equation for the Green's functions

$$-m \, \ddot{G}^{\pm}{}_{ij'} = - \delta_{ij} \, \delta(t-t') \quad . \qquad (9.4.5)$$

The solutions of Eq. (9.4.5) are readily found to be

$$
\left.
\begin{aligned}
G^{+}_{ij'} &= m^{-1}\,\delta_{ij}\,\theta(t'-t)\,(t'-t) \\
G^{-}_{ij'} &= m^{-1}\,\delta_{ij}\,\theta(t-t')\,(t-t')
\end{aligned}
\right\} \qquad (9.4.6)
$$

$\theta(t-t')$ being the step function for the present case. From this the fundamental Poisson bracket immediately follows:

$$
(x_i, x_{j'}) = G_{ij'} = -m\delta_{ij}(t-t') \quad , \qquad (9.4.7)
$$

leading to the uncertainity relation

$$
\Delta x_i\, \Delta x_{j'} \approx m^{-1}\,\delta_{ij}\,|t-t'| \;. \qquad (9.4.8)
$$

The physical interpretation of this uncertainty relation is immediately apparent. A measurement of x_i with accuracy Δx_i leads to an uncertainty in momentum of order $1/\Delta x_i$ and hence to an uncertainty in the velocity component $\dot{x}_i$ of order $m^{-1}/\Delta x_i$. This leads to a subsequent position uncertainty which increases with elapsed time, namely: $\Delta x_{i'} \sim (m^{-1}/\Delta x_i)|t-t'|$. The other components of position remain unaffected.

The entire quantum theory of the free particle can be based on the Poisson bracket (9.4.7) taken in the form of the commutator. The development follows completely familiar lines. We confine ourselves here to the derivation of the generator of infinitesimal displacements in time. The essential arguments have already been given at the end of the preceding section. Remembering that the change in the explicit form of the action which generates the displacement δt is equal to the <u>negative</u> of the variation in the action due to the variation in $\underset{\sim}{x}$ itself under this displacement we have

$$\delta S = -m\int \dot{x} \cdot d(\dot{x}\,\delta t) = m\int \ddot{x} \cdot \dot{x}\,\delta t\,dt = -\frac{1}{2}m\int \dot{x}^2\,\delta\dot{t}\,dt \qquad (9.4.9)$$

and hence

$$\dot{x}\,\delta t = \delta^{\pm} x = \begin{Bmatrix} \left(x, -\frac{1}{2}m\int_t^{\infty} \dot{x}'^2\,\delta\dot{t}'\,dt'\right) \\ \left(x, -\frac{1}{2}m\int_{-\infty}^{t} \dot{x}'^2\,\delta\dot{t}'\,dt'\right) \end{Bmatrix} = (x, H)\,\delta t \ , \qquad (9.4.10)$$

$$H = \frac{1}{2}m\,\dot{x}^2 \ . \qquad (9.4.11)$$

The final form is obtained through integration by parts and use of Eq. (9.4.4). The possible extra term Δx [cf. Eq.(9.3.60)] vanishes in the present case since $G_{ij'}\,\theta(t-t') = G_{ij'}\,\delta(t-t') = 0$. Furthermore, since the Hamiltonian H is constant in time, the equation

$$\dot{f} = (f, H) \qquad (9.4.12)$$

holds for quite general dynamical variables f.

The theory of the relativistic free particle can be developed in a quite similar fashion, starting from the action functional

$$S = -m\int (1-\dot{x}^2)^{\frac{1}{2}}\,dt \ . \qquad (9.4.13)$$

One obtains the equation for the Green's functions

$$-m(1-\dot{x}^2)^{-\frac{1}{2}}\,[\delta_{ik} + (1-\dot{x}^2)^{-1}\,\dot{x}_i\dot{x}_k]\,\ddot{G}^{\pm}{}_{kj'} = -\delta_{ij}\,\delta(t-t') \ , \qquad (9.4.14)$$

which leads to the Poisson bracket

$$(x_i, x_{j'}) = G_{ij'} = -m(1-\dot{x}^2)^{\frac{1}{2}}\,(\delta_{ij} - \dot{x}_i\dot{x}_j)(t-t'). \qquad (9.4.15)$$

In this form, however, the formalism is unsuitable for extension and application to the measurement problem in general relativity.

Since the dynamical equations derived from the action (9.4.13) are Lorentz covariant, the formalism itself should be made Lorentz covariant. What is needed is a manifestly Lorentz covariant Poisson bracket instead of one which, like (9.4.15), singles out the time for special treatment. For this purpose it is necessary to introduce the proper time. Proper time, however, must be reckoned starting from some zero point which the particle by itself is unable to provide. For example, it is not satisfactory to reckon proper time from the moment (i.e., space time event) when one of the spatial coordinates of the particle has a given value, or when the ordinary time itself has a given value. For such a reckoning would not treat all the space-time coordinates of the particle equally, and the resulting formalism would not, in fact, be manifestly covariant. In order to achieve a manifestly covariant formalism which is suitable for the measurement problem, one must have available an intrinsic proper time, and this can only be provided by a physical clock which "sits on" the particle.

For simplicity the clock itself may be regarded as being the particle. If the term "particle" is to remain applicable this means that the physical dimensions of the clock must be small, or else that the clock coordinates must be "internal" coordinates, unrelated to space-time. We do not concern ourselves here with the question of the practical realizability of such clocks. We must, however, inquire into the nature of the action functionals which describe them. For the sake of orientation it is convenient at this point to adopt the conventional Lagrangian - Hamiltonian viewpoint, although in the final development it will be dispensed with.

The internal dynamics of the clock will, in the rest frame, be describable by a Lagrangian $\ell(q, \dot{q})$ depending on a set of internal coordinates q^a and their time derivatives. Alternatively, the description may be made in terms of a Hamiltonian

$$m = p_a \dot{q}^a - \ell \ , \qquad p_a = \partial \ell / \partial \dot{q}^a \ , \qquad (9.4.16)$$

which is expressible as a function of q's and p's after the second of Eqs. (9.4.16) has been solved for the $\dot{q}$'s. The symbol m is used here for the Hamiltonian, since its value will be simply the rest mass of the clock (assuming proper choice of the energy zero point). In passing now to an arbitrary inertial frame it is only necessary to note that time derivatives become proper time derivatives. Referring back to coordinate time, therefore, we may write the Lagrangian in a general frame in the form

$$L = \ell(q, \dot{q}(1-\dot{\mathbf{x}}^2)^{-\frac{1}{2}})(1-\dot{\mathbf{x}}^2)^{\frac{1}{2}} \ . \qquad (9.4.17)$$

The momenta p_a remain unchanged in value, while the momenta conjugate to the x_i become

$$\begin{aligned} p_i &= \partial L/\partial \dot{x}_i = [p_a \dot{q}^a (1-\dot{\mathbf{x}}^2)^{-\frac{1}{2}} - \ell] \dot{x}_i (1-\dot{\mathbf{x}}^2)^{-\frac{1}{2}} \\ &= m \dot{x}_i (1-\dot{\mathbf{x}}^2)^{-\frac{1}{2}} \ . \qquad (9.4.18) \end{aligned}$$

The Hamiltonian therefore becomes

$$H = \mathbf{p} \cdot \dot{\mathbf{x}} + p_a \dot{q}^a - L = \mathbf{p} \cdot \dot{\mathbf{x}} + m(1-\dot{\mathbf{x}}^2)^{\frac{1}{2}} = (m^2 + \mathbf{p}^2)^{\frac{1}{2}} \ , \qquad (9.4.19)$$

just as for a particle without any internal degrees of freedom. All the internal variables are contained in the mass m.

Many different mechanical devices (including atoms and molecules!) are adaptable for use as clocks. In the majority of cases these are essentially conservative multiply periodic systems, and it will suffice to confine our attention to them. The description of multiply periodic systems is conveniently carried out with the aid of the classical angle and action variables. For purposes of measuring time only one of the angle variables is really necessary. Therefore it suffices to consider clocks having only one degree of freedom, with one action variable J and one angle variable θ. Other degrees of freedom may actually be present, but if we agree never to disturb them by measurements, the action variables associated with them (which determine the internal energy and hence the rest mass) will remain constant and may be ignored. The rest mass will then be a function of the single variable J, and the angular frequency of the internal motion will be given by

$$\omega = \partial m / \partial J \, . \tag{9.4.20}$$

The Hamiltonian equations for the clock become

$$\left.\begin{aligned} \dot{\underset{\sim}{x}} &= \underset{\sim}{p}(m^2 + \underset{\sim}{p}^2)^{\frac{1}{2}} \, , & \dot{\underset{\sim}{p}} &= 0 \, , \\ \dot{\theta} &= m(m^2 + \underset{\sim}{p}^2)^{-\frac{1}{2}} \omega \, , & \dot{J} &= 0 \, . \end{aligned}\right\} \tag{9.4.21}$$

The internal dynamical quantity $\omega^{-1}\Theta$ will be recognized as the intrinsic proper time. We may now ask such questions as: Where will the clock be when the intrinsic proper time has the numerical value τ, and what will the value of the coordinate time be at that instant? With the aid of the Hamiltonian equations it is easy to see that the answers to these questions are given respectively by

$$\underset{\sim}{x}(\tau) = \underset{\sim}{x} + m^{-1}\underset{\sim}{p}(\tau-\omega^{-1}\Theta) , \qquad (9.4.22)$$

$$t(\tau) = t + m^{-1}(m^2+\underset{\sim}{p}^2)^{\frac{1}{2}}(\tau-\omega^{-1}\Theta) , \qquad (9.4.23)$$

in terms of the values of the dynamical variables $\underset{\sim}{x}$, $\underset{\sim}{p}$, Θ at an arbitrary time t. Poisson brackets of $x(\tau)$ and $t(\tau)$ with each other may therefore be constructed in the conventional canonical manner; it is only necessary to bear in mind that m and ω are functions of J. A straightforward computation leads to the completely covariant result

$$(x^\mu(\tau), x^\nu(\tau')) = -m^{-1}(\eta^{\mu\nu}+\dot{x}^\mu\dot{x}^\nu)(\tau-\tau') , \qquad (9.4.24)$$

where the indices on the x_i have been raised, in accord with the introduction of the Minkowski metric $(\eta^{\mu\nu})$ = diag (-1,1,1,1) and where $x^0(\tau) \equiv t(\tau)$.

The dots are now used to denote differentiation with respect to the proper time. It will be observed that Eq. (9.4.24) is consistent with the identity $\dot{x}^\mu\dot{x}_\mu = -1$.

The covariant Poisson bracket above was obtained with non-covariant methods. We shall next see how the same result can be obtained with Green's functions within the framework of a manifestly covariant formalism. The simplest covariant description of the relativistic clock is provided by the action functional

$$S = \int [J\dot{\theta} - m(-\dot{x}^2)^{\frac{1}{2}}]dt \quad , \tag{9.4.25}$$

where t is here a completely arbitrary parameter,[13] differentiation with respect to which is denoted by the dot. (Note the varying uses of the dot!), and where $\dot{x}^2$ is an abbreviation for $\dot{x}^\mu \dot{x}_\mu$. The variational derivatives

$$\delta S/\delta J \equiv \dot{\theta} - \omega(-\dot{x}^2)^{\frac{1}{2}} = 0 \,, \tag{9.4.26}$$

$$\delta S/\delta\theta \equiv -\dot{J} = 0 \quad , \tag{9.4.27}$$

$$\delta S/\delta x^\mu \equiv -d(m\, v_\mu)/dt = m\, \dot{v}_\mu = 0 \,, \tag{9.4.28}$$

where

$$v^\mu \equiv \dot{x}^\mu(-\dot{x}^2)^{-\frac{1}{2}}, \qquad v^\mu v_\mu = -1 \,, \tag{9.4.29}$$

yield dynamical equations equivalent to those of (9.4.21). However, the second variational derivatives

$$\left.\begin{aligned}
\delta^2 S/\delta J\, \delta J' &\equiv -(\delta\omega/\delta J)(-\dot{x}^2)^{\frac{1}{2}}\, \delta(t,t') \quad , \\
\delta^2 S/\delta J\, \delta\theta' &\equiv \partial\, \delta(t,t')/\delta t\ (\\
\delta^2 S/\delta J\, \delta x^{\mu'} &\equiv \omega\, v_\mu\, \partial\, \delta(t,t')/\delta t, \\
\delta^2 S/\delta\theta\, \delta\theta' &\equiv 0, \\
\delta^2 S/\delta\theta\, \delta x^{\mu'} &\equiv 0, \\
\delta^2 S/\delta x^\mu\, \delta x^{\nu'} &\equiv -\partial[m(-\dot{x}^2)^{-\frac{1}{2}}\, P_{\mu\nu}\, \partial\, \delta(t,t')/\partial t]/\partial t
\end{aligned}\right\} \tag{9.4.30}$$

where

$$P_{\mu\nu} \equiv \eta_{\mu\nu} + v_\mu v_\nu \,, \qquad P_{\mu\nu} v^\nu \equiv 0 \,, \tag{9.4.31}$$

do not in the present case lead immediately to equations for the Green's functions of the system. This is because the action (9.4.25) possesses an infinite dimensional invariance group, namely that associated with the arbitrariness in the parameter t. Under the infinitesimal parameter transformation $t' = t - \delta t$ the dynamical variables suffer the changes

$$\left.\begin{aligned} \delta J &= \dot{J}\,\delta t = 0\ , \\ \delta\Theta &= \dot{\Theta}\,\delta t = \omega(-\dot{x}^2)^{\frac{1}{2}}\delta t, \\ \delta x^\mu &= \dot{x}^\mu\delta t = v^\mu(-\dot{x}^2)^{\frac{1}{2}}\,\delta t, \end{aligned}\right\} . \tag{9.4.32}$$

When the dynamical equations are satisfied the condition that a given dynamical quantity A be parameter invariant is evidently

$$\omega\frac{\delta A}{\delta\Theta} + v^\mu\frac{\delta A}{\delta x^\mu} = 0\ . \tag{9.4.33}$$

The action functional itself is, of course, parameter invariant.

In order to obtain definite solutions to the equations for the small disturbances $\delta^\pm J$, $\delta^\pm\Theta$, $\delta^\pm x^\mu$ induced in the system by the addition of an infinitesimal parameter invariant ϵA to the action(9.4.2), it is necessary to impose a supplementary condition. The one which is convenient here is

$$v_\mu\delta^\pm\dot{x}^\mu = 0\ . \tag{9.4.34}$$

If this condition is not already satisfied it can easily be imposed by first carrying out an infinitesimal parameter transformation (9.4.32) for which

$$\delta t = (-\dot{x}^2)^{-\frac{1}{2}}\int^t v_\mu\,\delta^\pm x^\mu\,dt\ . \tag{9.4.35}$$

When it is satisfied it is easily seen from Eqs.(9.4.30) and((9.4.31) that

$$\left.\begin{aligned}
\delta^{\pm}J &= \epsilon\int \left(G^{\pm}_{JJ'}\,\frac{\delta A}{\delta J'} + G^{\pm}_{J\Theta'}\,\frac{\delta A}{\delta\Theta'} + G^{\pm}{}_{J}{}^{\mu'}\,\frac{\delta A}{\delta x^{\mu'}}\right) dt' ,\\
\delta^{\pm}\Theta &= \epsilon\int \left(G^{\pm}_{\Theta J'}\,\frac{\delta A}{\delta J'} + G^{\pm}_{\Theta\Theta'}\,\frac{\delta A}{\delta\Theta'} + G^{\pm}{}_{\Theta}{}^{\mu'}\,\frac{\delta A}{\delta x^{\mu'}}\right) dt' ,\\
\delta^{\pm}x^{\mu} &= \epsilon\int \left(G^{\pm\mu}{}_{J'}\,\frac{\delta A}{\delta J'} + G^{\pm\mu}{}_{\Theta'}\,\frac{\delta A}{\delta\Theta'} + G^{\pm\mu\nu'}\,\frac{\delta A}{\delta x^{\nu'}}\right) dt' ,
\end{aligned}\right\} \quad (9.4.36)$$

where the $G^{\pm}$ are the Green's functions for the wave operator

$$\begin{pmatrix} F_{JJ'} & F_{J\Theta'} & F_{J\nu'} \\ F_{\Theta J'} & F_{\Theta\Theta'} & F_{\Theta\nu'} \\ F_{\mu J'} & F_{\mu\Theta'} & F_{\mu\nu'} \end{pmatrix}$$

$$\doteq \begin{pmatrix} -(\partial\omega/\partial J)(-\dot{x}^2)^{\frac{1}{2}}\delta(t,t') & \partial\delta(t,t')/\partial t & 0 \\ -\partial\delta(t,t')/\partial t & 0 & 0 \\ -\omega\, v_{\mu}\ \partial\,\delta(t,t')/\partial t & 0 & -m\eta_{\mu\nu}\partial[(-\dot{x}^2)^{-\frac{1}{2}}\partial\delta(t,t')/\partial t]/\partial t \end{pmatrix} . \quad (9.4.37)$$

The computation of the Green's functions is straightforward. One easily finds, for example, that the functions $G^{\pm}_{JJ'}$, $G^{\pm}{}_{J}{}^{\mu'}$, $G^{\pm}{}_{\Theta}{}^{\mu'}$, $G^{\pm\mu}{}_{J'}$ vanish. The Poisson bracket of two invariants, A and B therefore reduces to

$$(A,\ B) = \int dt \int dt' \left(\frac{\delta A}{\delta J} G_{J\Theta'}\,\frac{\delta B}{\delta\Theta'} + \frac{\delta A}{\delta\Theta}\,G_{\Theta J'}\,\frac{\delta B}{\delta J'} + \frac{\delta A}{\delta\Theta}\,G_{\Theta\Theta'}\,\frac{\delta B}{\delta\Theta'}\right.$$

$$\left. + \frac{\delta A}{\delta x^{\mu}}\,G^{\mu}{}_{\Theta'}\,\frac{\delta B}{\delta\Theta'} + \frac{\delta A}{\delta x^{\mu}}\,G^{\mu\nu'}\,\frac{\delta B}{\delta x^{\nu'}}\right) . \quad (9.4.38)$$

We postpone evaluating the remaining Green's functions until after we have introduced a description in terms of the proper time. First it is important to check that the solutions (9.4.36) satisfy the supplementary condition (9.4.34). From the equations

$$\left.\begin{aligned}
-\delta(t,t') &= \int (F_{\Theta J''}\, G^{\pm}{}_{J''\Theta'} + F_{\Theta\Theta''}\, G^{\pm}{}_{\Theta''\Theta'} + F_{\Theta\mu''}\, G^{\pm\mu''}{}_{\Theta'})\, dt'' \\
&= -\, \partial\, G^{\pm}{}_{J\Theta'}/\partial t \;, \\
0 &= \int (F_{\mu J''}\, G^{\pm}{}_{J''\Theta'} + F_{\mu\Theta''}\, G^{\pm}{}_{\Theta''\Theta'} + F_{\mu\nu''}\, G^{\pm\nu''}{}_{\Theta'})\, dt'' \\
&= -\, \omega\, v_\mu\, \partial\, G^{\pm}{}_{J\Theta'}/\partial t - m\, \partial\, [(-\dot{x}^2)^{\frac{1}{2}}\, \partial\, G^{\pm}{}_{\mu\Theta'}/\partial t\,]/\partial t, \\
-\delta_\mu{}^{\nu'} &= \int (F_{\mu J''}\, G^{\pm}{}_{J''}{}^{\nu'} + F_{\mu\Theta''}\, G^{\pm}{}_{\Theta''}{}^{\nu'} + F_{\mu\sigma''}\, G^{\pm\sigma''\nu'})\, dt'' \\
&= -\, m\, \partial\, [(-\dot{x}^2)^{\frac{1}{2}}\, \partial\, G^{\pm}{}_{\mu}{}^{\nu'}/\partial t\,]/\partial t \;,
\end{aligned}\right\} \qquad (9.4.39)$$

it follows that

$$\left.\begin{aligned}
v_\mu\, \partial\, G^{\pm\mu}{}_{\Theta'} / \partial t &= \omega\, G^{\pm}(t,\, t') \;, \\
v_\mu\, \partial\, G^{\pm\mu\nu'} / \partial t &= v^{\nu'}\, G^{\pm}(t,\, t'),
\end{aligned}\right\} \qquad (9.4.40)$$

where the functions $G^{\pm}$ satisfy

$$-\, m\, \partial[(-\dot{x}^2)^{\frac{1}{2}}\, G^{\pm}(t,\, t')\,]/\partial t \;=\; -\, \delta(t,\, t') \quad . \qquad (9.4.41)$$

We therefore have

$$v_\mu \delta^{\pm}\dot{x}^\mu \;=\; \epsilon \int G^{\pm}(t,t')\, (\omega' \frac{\delta A}{\delta\Theta'} + v^{\nu'} \frac{\delta A}{\delta x^{\nu'}})\, dt' \;, \qquad (9.4.42)$$

which vanishes on account of the invariance condition (9.4.33).

Consider now an arbitrary local function f of the dynamical variables and their t- derivatives. It is important to make a distinction between the quantity f, taken at an arbitrary value of t, and the same quantity taken at that value of t for which the intrinsic proper time $\omega^{-1}\Theta$ has the numerical value τ. For the moment we shall denote the latter quantity by f_τ. The relation between f_τ and f is

$$f_\tau \equiv \int (-\dot{x}^2)^{\frac{1}{2}} \delta(\tau - \omega^{-1}\Theta)\, f\, dt \, . \tag{9.4.43}$$

Using the dynamical equations it is straightforward to show that

$$\partial f_\tau / \partial\tau = [(-\dot{x}^2)^{-\frac{1}{2}} \dot{f}]_\tau \, . \tag{9.4.44}$$

Next, let us compute the disturbances in f_τ due to the addition of ϵA to the action. Making use of the supplementary condition (9.4.34) and the equations satisfied by $\delta^\pm J$ and $\delta^\pm\Theta$, as determined by the variational derivatives (9.4.30), we find, from Eq. (9.4.43),

$$\begin{aligned}\delta^\pm f_\tau \;=\; & (\delta^\pm f)_\tau + (\partial f_\tau/\partial\tau)\, \omega^{-1}[-(\delta^\pm\Theta)_\tau + \tau(\partial\omega/\partial J)(\delta^\pm J)_\tau] \\ & + \epsilon[(-\dot{x}^2)^{-\frac{1}{2}} f]_\tau\, \omega^{-1}[(\delta A/\delta J)_\tau + \tau(\partial\omega/\partial J)(\delta A/\delta\Theta)_\tau] \, .\end{aligned} \tag{9.4.45}$$

With the aid of this result we may reformulate the Poisson bracket of two invariants, A and B, in terms of intrinsic proper time. We first set t formally equal to τ, replace Θ by $\omega\tau$ wherever it occurs in the expressions for A and B, and regard A and B as explicit functionals of $(x^\mu)_\tau$ and J_τ. Then, noting that the second line of (9.4.45) disappears in the difference $\delta^+ f_\tau - \delta^- f_\tau$ which occurs in the definition of the Poisson bracket, we have

$$\begin{aligned}(A,B) \;=\; \int d\tau \int d\tau' \Big[& -\frac{\delta A}{\delta J} G_{J\Theta'}{v^{\mu'}}\, \omega^{-1} \frac{\delta B}{\delta x^{\mu'}} - \frac{\delta A}{\delta x^\mu} v^\mu\, \omega^{-1} G_{\Theta J'} \frac{\delta B}{\delta J'} \\ & + \frac{\delta A}{\delta x^\mu} (G^{\mu\nu'} - G^\mu{}_{\Theta'} v^{\nu'} \omega^{-1} + v^\mu \omega^{-1} G_{\Theta\Theta'} v^{\nu'} \omega^{-1} \\ & - v^\mu \omega^{-1} G_{\Theta J'} v^{\nu'} \omega^{-1} \tau' \frac{\partial\omega}{\partial J} - v^\mu \omega^{-1} \tau \frac{\partial\omega}{\partial J} G_{J\Theta'} v^{\nu'} \omega^{-1}) \frac{\delta B}{\delta x^{\nu'}} \Big]\end{aligned} \tag{9.4.46}$$

where the subscripts τ have now been dropped.

The equations for the Green's functions take the following forms in terms of the proper time:

$$\left.\begin{aligned}
\partial G^{\pm}_{\Theta J'}/\partial\tau &= -\,\partial G^{\pm}_{J\Theta'}/\partial\tau = -\delta(\tau - \ ') \,, \\
-\,(\partial\omega/\partial J)\, G^{\pm}_{J\Theta'} + \partial\, G^{\pm}_{\Theta\Theta'}/\partial\tau &= 0 \,, \\
-\,\omega\, v_{\mu}\, \partial G^{\pm}_{J\Theta'}/\partial\tau - m\, \partial^2 G^{\pm}_{\mu\Theta'}/\partial\tau^2 &= 0 \,, \\
-\,m\, \partial^2 G^{\pm}{}_{\mu}{}^{\nu'}/\partial\tau^2 &= -\delta_{\mu}{}^{\nu'} \,.
\end{aligned}\right\} \qquad (9.4.47)$$

The solutions of these equations are

$$\left.\begin{aligned}
G^{\pm}_{\Theta J'} &= -\,G^{\pm}_{J\Theta'} = \pm\,\theta(\mp(\tau - \tau')) \,, \\
G^{\pm}_{\Theta\Theta'} &= \mp\,(\partial\omega/\partial J)\, \theta(\mp\,(\tau - \tau'))(\tau - \tau') \,, \\
G^{\pm\mu}{}_{\Theta'} &= \pm\, m^{-1}\, \omega\, v^{\mu}\, \theta(\mp\,(\tau - \tau'))(\tau - \tau') \,, \\
G^{\pm\mu\nu'} &= \mp\, m^{-1}\, \eta^{\mu\nu}\, \theta(\mp\,(\tau - \tau'))(\tau - \tau') \,,
\end{aligned}\right\} \qquad (9.4.48)$$

and hence, finally,

$$(A,\ B) = \int d\tau \int d\tau' \left[\frac{\delta A}{\delta J}\, v^{\mu'}\, \omega^{-1}\, \frac{\delta B}{\delta x^{\mu'}} - \frac{\delta A}{\delta x^{\mu}}\, v^{\mu}\, \omega^{-1}\, \frac{\delta B}{\delta J'} - m^{-1}\, (\tau - \tau')\, \frac{\delta A}{\delta x^{\mu}}\, P^{\mu\nu}\, \frac{\delta B}{\delta x^{\nu'}} \right] . \qquad (9.4.49)$$

In particular,

$$(x^{\mu},\ x^{\nu'}) = -\, m^{-1}\, P^{\mu\nu}\, (\tau - \tau') \,, \qquad (9.4.50)$$

in agreement with Eq. (9.4.24).

In their application to the relativistic clock the Green's function techniques are evidently open to the criticism of being unduly complicated. The particular method which this simple example illustrates, however, is of great importance in the quantization program of general relativity and justifies the attention given to it. The process of passing from general to intrinsic coordinates will be encountered again in Section 6 and, in fact, will almost certainly be characteristic of any other covariant quantization procedure. It should also be pointed out that, by using the covariant procedure in the present example, we have gained somewhat more than the mere Poisson bracket (9.4.50). For example, the generator of infinitesimal displacements in proper time is readily derived from the action (9.4.25).

A displacement in proper time is described by a variation $\dot{x}^\mu \delta t$ in the space-time coordinates of the clock, with no variation in Θ or J. [Thus the variations (9.4.32) do not describe a proper time displacement; they correspond merely to a parameter transformation, in which the change in Θ effects a change in the correspondence between t and the intrinsic proper time which precisely cancels the proper time displacement effected by $\dot{x}^\mu \delta t$.] Remembering again the general rule that the explicit change δS in the action needed to effect the desired change in x^μ is equal to the negative of the variation in the action due to an explicit change in x^μ of the desired amount, we have

$$\delta S \equiv -\int m(-\dot{x}^2)^{-\frac{1}{2}} x_\mu \, d(\dot{x}^\mu \delta t) \equiv m \, \delta\dot{\tau} \, d\tau, \qquad (9.4.51)$$

in which t is set equal to the proper time in the second integral. Using the form (9.4.49) for the Poisson bracket, therefore, we have

$$\dot{x}\,\delta\tau = \delta^{\pm}x^{\mu} = \begin{Bmatrix} (x^{\mu}, \int_{\tau}^{\infty} m\, \delta\dot{\tau}' d\tau') \\ -(x^{\mu}, \int_{-\infty}^{\tau} m\, \delta\dot{\tau}' d\tau') \end{Bmatrix} = -(x^{\mu}, m)\,\delta\tau \quad , \tag{9.4.52}$$

in which the usual integration by parts has been performed and the τ-independence of m has been used. Dropping the $\delta\tau$, we have

$$\dot{x}^{\mu} = -(x^{\mu}, m) = v^{\mu} \quad , \tag{9.4.53}$$

and similarly

$$\dot{J} = -(J, m) = 0. \tag{9.4.54}$$

Equations (9.4.53) and (9.4.54) are verified at one by explicit use of (9.4.49).

The role of the rest mass as the generator of proper time displacements reflects its role as the variable conjugate to the intrinsic proper time $\omega^{-1}\Theta$. In the quantum theory this conjugate relationship has the consequence that a measurement of proper time with an accuracy $\Delta\tau$ implies an uncertainty in the rest mass of order $1/\Delta\tau$. Since mass is always positive this implies that the "classical" or "average" value of the clock's mass must be at least as big as $1/\Delta\tau$ in order that the measurement actually be possible. It should be pointed out in this connection that the angle variable, and hence the intrinsic proper time itself, has strict validity as a concept only in the classical limit. It is generally impossible, in any given case, to construct for a periodic system, an Hermitian operator which can be strictly regarded as a quantum proper time variable conjugate to the Hamiltonian of the rest frame; this is a consequence of the one-sided (positive) and discrete character of the rest-mass spectrum. Our theory of the relativistic clock, therefore, like our treatment of the elastic medium in the following sections, is essentially phenomenological. Nevertheless, the above estimate of the minimum mass required to effect a proper time measurement of given accuracy has a basic validity, as has been confirmed in studies of specific clock models by Salecker (1957).[14]

(9.5) The stiff elastic medium.

In order to use an elastic medium as a coordinate framework we must know something about its dynamical properties, so that we may be able to judge the accuracy of measurements made with its aid. Since the properties which determine this accuracy are mainly nonrelativistic ones, we look first at the nonrelativistic theory. This will then be followed by a description of the logical extension of the theory to the relativistic domain.

In the so-called "Lagrangian" scheme of coordinates the constituent particles of the medium are identified by a set of three labels u^a, $a = 1, 2, 3$. The u^a provide what is generally a curvilinearsystem of coordinates which changes with time according to the motion of the medium. Its "shape" is described with reference to a Cartesian inertial frame of so-called "Eulerian" coordinates x_i, $i = 1, 2, 3$. The relation between the Lagrangian and Eulerian coordinates is expressed by a set of three functions $x_i\ (t, \underset{\sim}{u})$ depending on the time t as well as the labels $\underset{\sim}{u} \equiv (u^1, u^2, u^3)$.

Characteristic properties of the medium may be expressed in either the Lagrangian or the Eulerian system. For example, if f is a density in the Eulerian system and f_0 is the corresponding density in the Lagrangian system, the relation between the two is given by

$$f = \frac{\partial(\underset{\sim}{u})}{\partial(\underset{\sim}{x})} f_0 \quad , \tag{9.5.1}$$

where $\partial(\underset{\sim}{u})/\partial(\underset{\sim}{x})$ is the Jacobian of the transformation from one system to the other. There are two principal densities which characterize an elastic medium: the mass density and the internal energy density, which we shall denote by ρ_0 and w_0 respectively. The mass density ρ_0 may vary from point to point in the medium, but will be otherwise constant; it depends, therefore, only on $\underset{\sim}{u}$. The constancy in time follows from the law of conservation of mass, which may be expressed in the form

$$\dot{\rho}_0 = 0 , \tag{9.5.2}$$

the dot denoting differentiation with respect to the time. More precisely, the dot will be used to denote partial time differeniation when the quantity over which it stands is regarded as a function of t and u. The same quantity may also be regarded as a function of t and $\underset{\sim}{x} \equiv (x_1, x_2, x_3)$, however, and in this case partial time differentiation will be denoted by a subscript t. The relation between the two kinds of derivatives, for an arbitrary quantity q, is

$$\dot{q} = q_t + q_{,i} v_i , \tag{9.5.3}$$

$$v_i \equiv \dot{x}_i \equiv \partial x_i(t, \underset{\sim}{u})/\partial t , \tag{9.5.4}$$

the comma followed by an index denoting differentiation with respect to x_i. From this relation together with the law of differentiation of Jacobians, the mass conservation law in the Eulerian system is readily derived:

$$\rho_t + (\rho v_i)_{,i} = 0 . \tag{9.5.5}$$

The internal energy density w_0, like ρ_0, may vary from point to point in the medium. Its variations, however, unlike those of ρ_0, are not "pre-set" but depend in some measure at least on the dynamical situation. It will be assumed that the spatial dependence of w_0 can be cleanly separated into two well defined parts, (1) an explicit dependence on $\underset{\sim}{u}$, reflecting a possible spatial variation in the basic constitution of the medium, and (2) an explicit dependence on the shape of the Lagrangian coordinate mesh at the point $\underset{\sim}{u}$. The shape of the Lagrangian mesh is described by the Lagrangian metric

$$\gamma_{ab} \equiv x_{i,a}\, x_{i,b} \quad , \tag{9.5.6}$$

(Here the comma followed by a lower case Latin index from the beginning of the alphabet denotes differentiation with respect to a u.) The dependence of w_0 on the time enters only through its dependence on γ_{ab}.

The use of ρ_0 and w_0 constitutes a phenomenological description of the medium, the validity of which depends on the adequacy with which gross properties of its actual atomic, colloidal or granular structure can be treated by means of instantaneous averages. The phenomenological description has been well established, on the nonrelativistic level, for many experimentally analyzed solid materials. However, our program here, being of a conceptual nature, does not hang on actual laboratory observations, and we shall, in fact, assume the validity of the phenomenological description in the relativistic domain as well. Furthermore, in the relativistic extension of the theory we shall assume that we can continue to maintain a sharp distinction between ρ_0 and w_0. The quantity ρ_0 will be regarded as the rest energy density due to the masses of the constituent particles while w_0 is regarded as the rest energy density arising from internal stresses, i.e., interactions between the particles.

The form of the action functional for the elastic medium may be inferred from nonrelativistic particle mechanics. In the Lagrangian system it is given by

$$S \quad \int dt \int d^3\underset{\sim}{y}\, \left(\frac{1}{2}\rho_0 \dot{x}_i \dot{x}_i - w_0\right) . \tag{9.5.7}$$

The dynamical equations are

$$\delta S/\delta x_i \equiv - \rho_0 \ddot{x}_i - (t^{ab} x_{i,a})_{,b} = 0, \tag{9.5.8}$$

where

$$t^{ab} \equiv -2\, \partial w_0/\partial \gamma_{ab} . \tag{9.5.9}$$

In the Eulerian system these equations take the form

$$-\rho\, (v_{it} + v_{i,j}\, v_j) - t_{ij,j} = 0 , \tag{9.5.10}$$

$$t_{ij} \equiv \frac{\partial(\underset{\sim}{y})}{\partial(\underset{\sim}{x})}\, x_{i,a} x_{j,b} t^{ab} , \tag{9.5.11}$$

which identifies t_{ij} as the internal stress density, giving rise to a body force density of amount

$$f_i \equiv - t_{ij,j} . \tag{9.5.12}$$

This identification is also confirmed through a consideration of the work done by this force <u>on</u> the constituent particles of the medium under an infinitesimal displacement δx_i :

$$\delta w = \int f_i \delta x_i \, d^3x = \int t_{ij} \delta x_{i,j} \, d^3\underset{\sim}{x} = \int t^{ab} x_{i,a} \delta x_{i,b} \, d^3\underset{\sim}{u}$$

$$= -\int \frac{\partial w_0}{\partial \gamma_{ab}} \delta\gamma_{ab} \, d^3\underset{\sim}{u} = -\int \delta w_0 \, d^3\underset{\sim}{u} \quad . \tag{9.5.13}$$

The energy supplied to perform this work must come from the medium itself and therefore shows up as a loss in the internal energy.

In combination with the mass conservation law, Eq. (9.5.10) may be re-expressed in the form of a momentum conservation law:

$$(\rho v_i)_t + T_{ij,j} = 0 \quad , \tag{9.5.14}$$

$$T_{ij} = \rho \, v_i v_j + t_{ij} \quad . \tag{9.5.15}$$

T_{ij} is the stress-momentum-flux density. With the aid of the identity

$$\dot{w}_0 = -\frac{\partial(\underset{\sim}{x})}{\partial(\underset{\sim}{u})} v_{i,j} \, t_{ij} \quad , \tag{9.5.16}$$

Eqs. (9.5.5) and (9.5.14) may also be combined into a law of energy conservation:

$$(\tfrac{1}{2} \rho v_i v_i + w)_t + (\tfrac{1}{2} \rho v_i v_i v_j + w v_j + v_i t_{ij})_{,j} = 0 \tag{9.5.17}$$

The changes in the dynamical variables x_i under the addition of an infinitesimal ϵA to the action (9.5.8) satisfy the equation

$$- \rho_0 \delta^{\pm} \ddot{x}_i + (c^{abcd} x_{i,a} x_{j,c} \delta^{\pm} x_{j,d})$$

$$- (t^{ab} \delta^{\pm} x_{i,a})_{,b} = - \epsilon \delta A / \delta x_i \quad , \tag{9.5.18}$$

$$c^{abcd} \equiv 4 \frac{\partial^2 w_0}{\partial \gamma_{ab} \partial \gamma_{cd}} . \tag{9.5.19}$$

There are no infinite dimensional invariance groups for this system, and hence the Poisson bracket may be written at once in the form

$$(A, B) = \int dt \int d^3\underset{\sim}{u} \int dt' \int d^3\underset{\sim}{u}' \frac{\delta A}{\delta x_i} G_{ij'} \frac{\delta B}{\delta x_{j'}} , \tag{9.5.20}$$

where $G_{ij'}$ is the propagation function formed from Green's functions satisfying the equation

$$- \rho_0 \ddot{G}^{\pm}_{ij'} + (c^{abcd} x_{i,a} x_{k,c} G^{\pm}_{kj',d})_{,b} - (t^{ab} G^{\pm}_{ij',a})_{,b} = -\delta_{ij'} . \tag{9.5.21}$$

The generator of infinitesimal displacements in time may be obtained in the now familiar manner. The pertinent variation in the explicit form of the action is

$$\begin{aligned} \delta S &\equiv - \int dt \int d^3\underset{\sim}{u} \, [\rho_0 \dot{x}_i \partial(\dot{x}_i \delta t)/\partial t + t^{ab} x_{i,a} \dot{x}_{i,b} \delta t] \\ &\equiv - \int dt \int d^3\underset{\sim}{u} \, (\tfrac{1}{2} \rho_0 \dot{x}_i \dot{x}_i + w_0) \, \delta\dot{t} , \end{aligned} \tag{9.5.22}$$

whence

$$\dot{f} = (f, H) , \tag{9.5.23}$$

$$H \equiv \int (\tfrac{1}{2} \rho_0 \dot{x}_i \dot{x}_i + w_0) \, d^3\underset{\sim}{u} . \tag{9.5.24}$$

Here, because of the constancy of the Hamiltonian H, Eq. (9.5.23) holds for an arbitrary dynamical variable f. The generator of infinitesimal spatial displacements may be similarly obtained. In this case the pertinent variation is

$$\delta S \equiv - \int dt \int d^3\underset{\sim}{u} \, [\rho_0 \dot{x}_i \dot{x}_{i,a} \delta u^a + t^{ab} x_{i,a} (x_{i,c} \delta u^c)_{,b}] . \tag{9.5.25}$$

whence

$$x_{i,a}\delta u^a = \delta^{\pm} x_i$$

$$= \begin{array}{l} x_i' \;,\; -\int_t^\infty dt' \int d^3\underset{\sim}{u}' \,[\rho_0' \dot{x}_{j'} \delta^{\pm}\dot{x}_{j'} - (t^{a'b'} x_{j',a'})_{,b'} \delta^{\pm} x_{j'}]) \\ -(x_i \;,\; -\int_\infty^t dt' \int d^3\underset{\sim}{u}' \,[\rho_0' \dot{x}_{j'} \delta^{\pm}\dot{x}_{j'} - (t^{a'b'} x_{j',a})_{,b'} \delta^{\pm} x_{j'}]) \end{array}$$

$$= \quad \left(x_i \;, \int \rho_0' x_{j'} \delta^{\pm} x_{j'} \, d^3\underset{\sim}{u}'\right)_{t'=t} \quad . \tag{9.5.26}$$

The possible extra term Δx_i [cf. Eq. (9.3.60)] vanishes since $G_{ij',a}\,\theta(t - t') = 0$, as may be inferred from the fact that the velocity of propagation of small disturbances relative to the medium is either zero (absence of internal stresses) or finite, so that $G_{ij'} = 0$ when $t = t'$. Since the total momentum

$$P_i \equiv \int \rho_0 \, \dot{x}_i \, d^3u \equiv \int \rho \, v_i \, d^3x \tag{9.5.27}$$

is conserved, Eq. (9.5.26) may immediately be generalized to

$$f_{,i} = (f, P_i) \tag{9.5.28}$$

for all f. It is to be noted that spatial displacements are treated separately from displacements in time in this nonrelativistic theory; thus the δu^a have no dependence on t, and δt has no dependence on the u^a.

The defining equation for the Green's function has an undesirable feature when written in the form (9.5.21), namely, some of its indices refer to the Lagrangian system while others refer to the Eulerian system. For many purposes it is convenient to transform completely to the Lagrangian system. A vector A_i

in the Eulerian system is transformed into a contravariant vector A^a in the Lagrangian by

$$A^a = u^a_{,i} A_i \quad , \tag{9.5.29}$$

and, reciprocally,

$$A_i = x_{i,a} A^a \quad . \tag{9.5.30}$$

Furthermore,

$$A^a_{\cdot b} = u^a_{,i} A_{i,j} x_{j,b} = u^a_{,i} A_{i,b} \quad , \tag{9.5.31}$$

where the dot followed by an index denotes covariant differentiation with respect to the Lagrangian metric γ_{ab}. It is important to note that the transformation coefficients $x_{i,a}$ and their reciprocals $u^a_{,i}$ generally depend on the time. Therefore, when time derivatives are performed the velocity and absolute acceleration of the medium relative to the Eulerian inertial frame make their appearance. Thus, defining

$$v^a \equiv u^a_{,i} \dot{x}_i \quad , \qquad a^a \equiv u^a_{,i} \ddot{x}_i \quad , \tag{9.5.32}$$

and differentiating Eq. (9.5.30), we find

$$u^a_{,i} \ddot{A}_i = \ddot{A}^a + 2v^a_{\cdot b} \dot{A}^b + a^a_{\cdot b} A^b \quad . \tag{9.5.33}$$

The Green's function equation in the Lagrangian system therefore takes the following form:

$$-\rho_0 (\ddot{G}^{\pm a}{}_{b'} + 2v^a{}_{.c}\, \dot{G}^{\pm c}{}_{b'} + a^a{}_{.c}\, G^{\pm c}{}_{b'})$$

$$+ (c^{acde}\, G^{\pm}{}_{db'.e})_{.c} - (t^{cd}\, G^{\pm a}{}_{b'.c})_{.d} = -\delta^a{}_{b'} \quad . \quad (9.5.34)$$

The elastic medium is most useful in providing a "laboratory coordinate system" when it is in a "ground state," in which its oscillation modes are quiescent and its bulk motion is uniform rectilinear. This dynamical condition is described by the vanishing of the internal stresses t^{ab} as well as of the quantities $v^a{}_{.b}$ and a^a. It must be recognized, however, that such a description is a semi-classical one, and an investigation must be made of its consistency with the actual presence of zero point quantum fluctuations. The quantum fluctuations may be described interms of the difference between the actual Eulerian positions x_i of the constituent particles of the medium and the average values $\langle x_i \rangle$ of these positions:

$$\delta x_i \equiv x_i - \langle x_i \rangle \qquad (9.5.35)$$

The validity of the semi-classical approximation depends on the accuracy with which average values of products may be replaced by products of average values. It will, in particular, depend on the accuracy with which the average value of the Lagrangian metric may be expressed in the form

$$\langle \gamma_{ab} \rangle = \langle x_{i,a} \rangle \langle x_{i,b} \rangle \qquad (9.5.36)$$

The actual Lagrangian metric is given by

$$\gamma_{ab} = \langle x_{i,a} \rangle \langle x_{i,b} \rangle + 2\, s_{ab} + \delta\, x_{c.a}\, \delta\, x^c{}_{.b} \;, \qquad (9.5.37)$$

where s_{ab} is the strain tensor:

$$s_{ab} \equiv \frac{1}{2}(\delta x_{a\cdot b} + \delta x_{b\cdot a}) \quad , \tag{9.5.38}$$

the covariant derivatives being defined with respect to the metric (9.5.36) and the passage to the Lagrangian system being now understood as effected by the transformation coefficients $\langle x_{i,a}\rangle$. The problem therefore becomes one of determining the conditions under which the mean value of the product $\delta x_{c\cdot a}\, \delta x^{c}{}_{\cdot b}$ will be small compared to $\langle \gamma_{ab}\rangle$. Our approach will be to begin by assuming the semi-classical approximation and then to demand that it be self-consistent.

The smallness of the product $\delta x_{c\cdot a}\, \delta x^{c}{}_{\cdot b}$ implies that the strain tensor itself is effectively small. A Hooke's Law approximation to the elastic forces may therefore be assumed, and, by appropriate adjustment of the zero point, the internal energy may be expressed in the form

$$w_0 = \frac{1}{2} c^{abcd} s_{ab} s_{cd} \quad , \tag{9.5.39}$$

yielding for the stress density the expression

$$t^{ab} = - c^{abcd} s_{cd} \quad , \tag{9.5.40}$$

where the c^{abcd} are now independent of the dynamical state and depend only on $\underset{\sim}{\mu}$. These expressions depend, of course, on the original assumption that the medium is stressless, on the average, in the quantum state in question:

$$\langle s_{ab} \rangle = 0 , \qquad \langle t^{ab} \rangle = 0 . \tag{9.5.41}$$

It is convenient to choose the Lagrangian system in such a way that it becomes Cartesian when the stresses vanish. That is,

$$\langle \gamma_{ab} \rangle = \delta_{ab} . \tag{9.5.42}$$

All Lagrangian indices may then be written in the lower position, and covariant derivatives become ordinary derivatives. The "displacement vactors" δx_a satisfy the commutation relation

$$[\delta x_a , \delta x_{b'}] = i G_{ab'} , \tag{9.5.43}$$

where $G_{ab'}$ is the propagation function formed from the Green's functions of equation (9.5.34) which, in virtue of Eq. (9.5.41) and the conditions of uniform motion, namely,

$$\langle v_{a,b} \rangle = 0 , \qquad \langle a_a \rangle = 0 , \tag{9.5.44}$$

here reduces to

$$- \rho_0 \ddot{G}^{\pm}_{ab'} + (c_{acde} G^{\pm}_{db',e})_{,c} = \delta_{ab'} . \tag{9.5.45}$$

It will suffice now to restrict the discussion to the case of a uniform isotropic medium. The condition of uniformity means that the "elastic moduli" c_{abcd} are simple constants, independent of $\underset{\sim}{u}$, while the condition of isotropy means that they can depend only on the Kronecker delta. The most general expression having the symmetries of c_{abcd} is

$$c_{abcd} = \lambda \delta_{ab} \delta_{cd} + \mu (\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}), \tag{9.5.46}$$

where λ is the Lamé constant and μ is the shear modulus.[15] Equation (9.5.45) therefore becomes

$$-\rho_0\,\ddot{G}^{\pm}_{ab'} + (\lambda + \mu)\,G^{\pm}_{cb',ca} + \mu\,G^{\pm}_{ab',cc} = -\,\delta_{ab'}\,, \qquad (9.5.47)$$

which can be solved by tha standard methods of field theory. Thus, introducing dyadic notation and making use of the Fourier decomposition of the delta function, one immediately finds

$$\underset{\sim}{G}^{\pm} = (2\pi)^{-4}\int d^3\underset{\sim}{k}\int_{C^{\pm}} d\omega\,[(\underset{\sim}{e}_1\,\underset{\sim}{e}_1 + \underset{\sim}{e}_2\,\underset{\sim}{e}_2)\,f(k)$$
$$+\,\underset{\sim}{e}_3\,\underset{\sim}{e}_3\,g(h)]\,e^{i[\underset{\sim}{k}\cdot(\underset{\sim}{u}-\underset{\sim}{u}')-\omega(t-t')]}\,, \qquad (9.5.48)$$

where $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, $\underset{\sim}{e}_3$ are mutually orthogonal unit vectors with $\underset{\sim}{e}_3 = \underset{\sim}{k}/k$, $k \equiv |\underset{\sim}{k}|$, and where

$$f(k) = -\frac{1}{\rho_0}\,\frac{1}{\omega^2 - c_t^{\,2}k^2}\,, \qquad g(k) = -\frac{1}{\rho_0}\,\frac{1}{\omega^2 - c_\ell^{\,2}k^2}\,, \qquad (9.5.49)$$

$$c_t = \sqrt{\mu/\rho_0}\,, \qquad c_\ell = \sqrt{(\lambda + 2\mu)/\rho_0}\,. \qquad (9.5.50)$$

The contours $C^{\pm}$ in the complex ω-plane are shown in Fig. (9-1). The evaluation of the integrals is straightforward, and one finds

$$\underset{\sim}{G}^{\pm} = \frac{1}{4\pi}\Bigg[\frac{1}{\mu}(\underset{\sim}{1} - \underset{\sim}{\nabla}\,\nabla^{-2}\underset{\sim}{\nabla})\,\frac{1}{|\underset{\sim}{u}-\underset{\sim}{u}'|}\,\delta(t - t' \pm \frac{1}{c_t}|\underset{\sim}{u}-\underset{\sim}{u}'|)$$
$$+\,\frac{1}{\lambda+2\mu}\,\underset{\sim}{\nabla}\,\nabla^{-2}\underset{\sim}{\nabla}\,\frac{1}{|\underset{\sim}{u}-\underset{\sim}{u}'|}\,\delta(t - t' \pm \frac{1}{c_\ell}|\underset{\sim}{u}-\underset{\sim}{u}'|)\Bigg]\,, \qquad (9.5.51)$$

where 1 is the unit dyadic and ∇^{-2} is an abbreviation for the integral operation involving, as a kernel, the Green's function $(4\pi)^{-1}|\underset{\sim}{u} - \underset{\sim}{u}'|^{-1}$ of the Laplacian operator. The quantities c_t

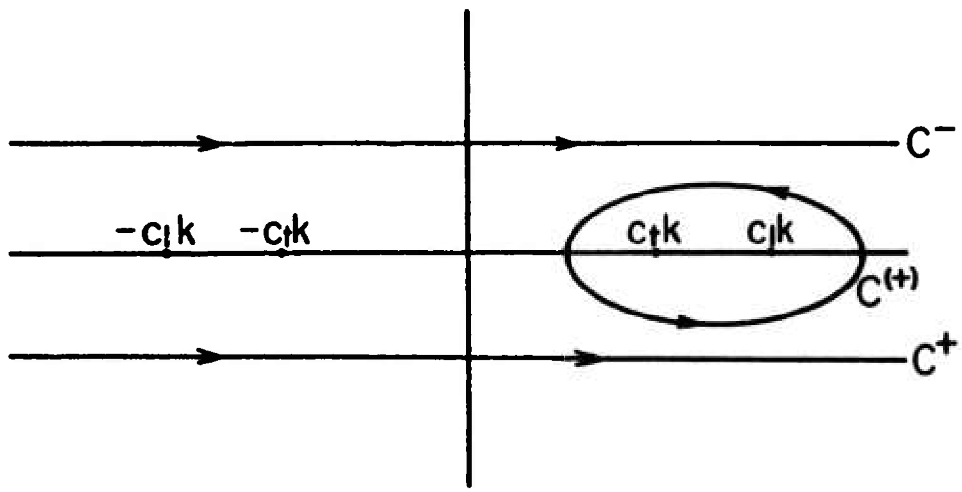

Fig. 9-1. Contours for the Green's functions of the elastic medium.

and c_ℓ are identifiable as the transverse and longitudinal sound velocities respectively.

The definition of the "ground state," which we may denote by $|0\rangle$, is expressible in the usual way in terms of the positive and negative frequency components of the displacement vector δx_a:[16]

$$\delta x_a^{(+)} |0\rangle = 0 , \qquad \langle 0| \delta x_a^{(-)} = 0 . \tag{9.5.52}$$

From this it follows that the mean value of the product $\delta x_a \delta x_{b'}$ in the ground state is given by

$$\begin{aligned} \langle \delta x_a \, \delta x_{b'} \rangle &= \langle (\delta x_a^{(+)} + \delta x_a^{(-)}) \, \delta x_{b'} \rangle = \langle \delta x_a^{(+)} \, \delta x_{b'} \rangle \\ &= \langle [\delta x_a^{(+)} , \delta x_{b'}] \rangle = i \, G^{(+)}_{ab'} , \end{aligned} \tag{9.5.53}$$

where $G^{(+)}_{ab'}$ is the positive frequency component of the propagation function, the integral representation of which is identical with (9.5.48) but with the contours $C^\pm$ replaced by the contour $C^{(+)}$ of Fig. (9-1). Contracting Eq. (9.5.53), differentiating with respect to $\underset{\sim}{u}$ and $\underset{\sim}{u}'$, and then setting $\underset{\sim}{u} = \underset{\sim}{u}'$, $t = t'$, we readily find, using the integral representation,

$$\begin{aligned} \langle \delta x_{c,a} \, \delta x_{c,b} \rangle &= \frac{1}{(2\pi)^3} \frac{1}{\rho_0} \int k_a k_b \left(\frac{1}{c_t k} + \frac{1}{2c_\ell k} \right) d^3\underset{\sim}{k} \\ &= \frac{\delta_{ab}}{3(2\pi)^3} \frac{1}{\rho_0} \left(\frac{1}{c_t} + \frac{1}{2c_\ell} \right) \int k \, d^3\underset{\sim}{k} . \end{aligned} \tag{9.5.54}$$

The quartic divergence of the final expression represents a breakdown in the continuum description of the medium. An actual elastic medium will be composed of a large number of particles, all of which may for simplicity be assumed to have the same mass m. The density is determined by the mean interparticle separation ℓ,

$$\rho_0 = m\,\ell^{-3} \quad , \tag{9.5.55}$$

and the continuum description becomes invalid for wavelengths shorter than this distance. By counting the number of degrees of freedom in the medium one sees, in fact that the interparticle separation provides an effective cut-off for the integral (9.5.54). For purposes of making estimates of orders of magnitude this cut-off may be taken as

$$k_{max} \simeq \frac{2\pi}{\ell} \quad . \tag{9.5.56}$$

We then have

$$\langle \delta x_{c,a}\, \delta x_{c,b} \rangle = \frac{2}{3}\pi^2 \frac{1}{m\ell}\left(\frac{1}{c_t} + \frac{1}{2c_\ell}\right)\delta_{ab} \, , \tag{9.5.57}$$

and the condition that this average be small compared with the metric (9.5.42) is evidently

$$\frac{2}{3}\pi^2 \frac{1}{m\ell}\left(\frac{1}{c_t} + \frac{1}{c_\ell}\right) \ll 1 \quad . \tag{9.5.58}$$

This is also essentially the condition for which the quantum fluctuations in the positions of the constituent particles remain small compared to the interparticle separation:

$$\langle \delta x_a\, \delta x_a \rangle \ll \ell^2 \quad . \tag{9.5.59}$$

Since the integral (9.5.54) is heavily weighted toward the cut-off end, it is clear that Eq. (9.5.57) provides a good estimate of the local fluctuations in the Lagrangian metric even when the density and elastic moduli vary from place to place in the medium, provided only that the variation is small over a distance ℓ (which is in any case required for the validity of the continuum description). The estimate is also good for media which are confined to limited regions of space (having dimensions, of course, large compared to ℓ). For the behavior of the interior of the medium will be relatively insensitive to "edge effects," and the long wavelength end of the "phonon" spectrum may be adequately treated by the imposition of periodic boundary conditions. This fact is important since, in the analysis of the measurability of the gravitational field, it permits us to limit the introduction of physical coordinate frames to particular regions of interest, so that all of space will not have to be filled with an elastic medium.

In the extension of the theory to the relativistic domain the additional requirements

$$c_t < 1 , \qquad c_\ell < 1 , \tag{9.5.60}$$

must be imposed. From this we infer

$$m\ell \gg \pi^2 \qquad \text{or} \qquad \ell \gg \frac{\pi^2}{m} > \frac{2\pi}{m} \tag{9.5.61}$$

which says that the interparticle separation must be large compared to the Compton wavelength of the particles if condition (9.5.58) is to hold. It has been emphasized by Pauli (1921) that the conditions (9.5.58) must not be regarded as implying that there is an absolute upper limit to the values of the elastic moduli which a medium can possess. The principle of relativity can say nothing about the possible strengths of interparticle forces. It can only say that if the static moduli become too large then the above phenomenological description of the medium-in-motion must break down. A dispersion of elastic waves must occur, and the group velocities will satisfy the conditions (9.5.60).

The conditions (9.5.60) and (9.5.61) have an important consequence for the magnitude of the contribution which the zero point fluctuations make to the total energy density of the medium. This contribution is easily calculated from the usual sum over elementary oscillators:

$$
\begin{aligned}
(2\pi)^{-3} \int_{k < k_{max}} \tfrac{1}{2} \left(2c_t k + c_\ell k \right) d^3\underset{\sim}{k} &= \pi^2 \ell^{-4} \left(2c_t + c_\ell \right) \\
&< 3\, \pi^2 \ell^{-4} \ll m \ell^{-3} = \rho_0 \, , \qquad (9.5.62)
\end{aligned}
$$

It is seen to be negligible compared to the rest energy of the medium,

The formal mathematics required to place the theory of the elastic medium in the context of special relativity has been developed by Herglotz (1911). Its extension to general Riemannian space-times with fixed metric is straightforward. We retain the labels u^a for the constituent particles but now describe their motion in terms of world lines given by a set of four functions

$x^{\mu}(t, \underset{\sim}{u})$, where t is an arbitrary parameter.[13] The symbols x^{μ} refer to a completely arbitrary set of curvilinear coordinates in space-time. Commas will denote ordinary differentiation and dots covariant differentiation with respect to the space-time metric $g^{\mu\nu}$. Differentiation with respect to the parameter t will be denoted by an overhead dot and will be performed only on quantities which are regarded as functions of t and $\underset{\sim}{u}$.

The Lagrangian metric γ_{ab} which determines the internal energy density must now be determined through strictly local considerations. For this purpose it is convenient to introduce the unit velocity field

$$v^{\mu} = (-\dot{x}^2)^{-\frac{1}{2}} \dot{x}^{\mu} , \qquad v_{\mu}v^{\mu} = -1 , \tag{9.5.63}$$

together with three other unit vector fields $n_i{}^{\mu}$, i = 1, 2, 3, satisfying

$$n_{i\mu} v^{\mu} = 0 , \qquad n_{i\mu} n_j{}^{\mu} = \delta_{ij} , \tag{9.5.64}$$

and hence

$$n_i{}^{\mu} n_i{}^{\nu} = P^{\mu\nu} \equiv g^{\mu\nu} + v^{\mu} v^{\nu} , \tag{9.5.65}$$

The vectors $n_i{}^{\mu}$ define a three dimensional local Cartesian rest frame at each point in the medium. A displacement δx_i with respect to this frame corresponds to a displacement of the coordinates x^{μ} of amount

$$\delta x^{\mu} = n_i{}^{\mu} \delta x_i , \tag{9.5.66}$$

and to displacements of the t, u^a by amounts δt, δu^a satisfying

$$\delta x^\mu = \dot{x}^\mu \delta t + x^\mu{}_{,a} \delta u^a \quad , \qquad \delta t = (-\dot{x}^2)^{-\frac{1}{2}} v_\mu x^\mu{}_{,a} \delta u^a , \quad (9.5.67)$$

whence

$$\delta x_i = n_{i\mu} x^\mu{}_{,a} \delta u^a \quad , \qquad (9.5.68)$$

$$\delta x_i \delta x_i = \gamma_{ab} \delta u^a \delta u^b \quad , \qquad (9.5.69)$$

$$\gamma_{ab} = P_{u\nu} x^\mu{}_{,a} x^\nu{}_{,b} \; . \qquad (9.5.70)$$

Although the Lagrangian metric defined in this way describes the local deformation of the medium as viewed in the instantaneous rest frame, it will not generally be the metric of any actual hypersurface. Since the quantity $v_u x^\mu{}_{,a}$ generally does not vanish the displacement δx_i will usually involve a displacement in the parameter t [see Eq. (9.5.67)].

The rest-densities of mass and internal energy in the Lagrangian system are assumed to be the same functions ρ_0 and w_0, as before. The corresponding densities in the local Cartesian frame are obtained through multiplication by the inverse of the determinant of the transformation coefficients $n_{i\mu}x^\mu{}_{,a}$. This determinant is evidently equal to the square root of the determinant

$$\gamma \equiv \det (\gamma_{ab}) \qquad (9.5.71)$$

of the Lagrangian metric. Making use of the identity [17]

$$\epsilon_{\mu\nu\sigma\tau} v^{\mu} \equiv g^{-\frac{1}{2}} \epsilon_{ijk}\, n_{i\nu}\, n_{j\sigma}\, n_{k\tau} , \qquad g \equiv -\det(g_{\mu\nu}) , \tag{9.5.72}$$

where ϵ_{ijk} and $\epsilon_{\mu\nu\sigma\tau}$ are respectively the three and four dimensional antisymmetric permutation symbols, one easily finds that the determinant of the transformation coefficients may also be expressed in the form

$$\gamma^{\frac{1}{2}} = g^{\frac{1}{2}} (-\dot{x}^{2})^{-\frac{1}{2}} \frac{\partial(x)}{\partial(t,u)} . \tag{9.5.73}$$

To obtain densities in the general coordinate frame a further multiplication must be performed by the determinant of the coefficients of the transformation from the local Lorentz frame, defined by combining the $n_i{}^{\mu}$ with v^{μ}, to the local mesh formed by the x^{μ}. This latter determinant is just $g^{\frac{1}{2}}$. Therefore the relation between a rest density f_0 in the Lagrangian system and the corresponding "proper density" f in the general coordinate frame is given by

$$f = g^{\frac{1}{2}} \gamma^{-\frac{1}{2}} f_0 = (-\dot{x}^{2})^{\frac{1}{2}} \frac{\partial(t,u)}{\partial(x)} f_0 . \tag{9.5.74}$$

The internal stress density may likewise be defined in the various reference systems. We have

$$t_{ij} = \gamma^{-\frac{1}{2}} n_{i\mu}\, x^{\mu}{}_{,a}\, n_{j\nu}\, x^{\nu}{}_{,b}\, t^{ab} ,$$

$$t^{\mu\nu} = g^{\frac{1}{2}} n_i{}^{\mu} n_j{}^{\nu} t_{ij} = (-\dot{x}^{2})^{\frac{1}{2}} \frac{\partial(t,u)}{\partial(x)} P^{\mu}{}_{\sigma} P^{\nu}{}_{\tau} x^{\sigma}{}_{,a} x^{\tau}{}_{,b} t^{ab} , \tag{9.5.75}$$

where t^{ab} is given by Eq. (9.5.9) as before. Similarly we define

$$c^{\mu\nu\sigma\tau} \equiv (-\dot{x}^2)^{\frac{1}{2}} \frac{\partial(t,u)}{\partial(x)} P^\mu{}_\iota P^\nu{}_\kappa P^\sigma{}_\chi P^\pi{}_\rho x^\iota{}_{,a} x^\kappa{}_{,b} x^\lambda{}_{c} x^\rho{}_{c} c^{abcd}. \qquad (9.5.76)$$

We note that

$$t^{\mu\nu} v_\nu = 0, \qquad c^{\mu\nu\sigma\tau} v_\tau = 0 . \qquad (9.5.77)$$

Conservation of mass (in this case rest-mass) is again expressed by Eq. (9.5.2). It is easily verified that in the general coordinate frame this becomes

$$(\rho v^\mu)_{\cdot\mu} = 0 . \qquad (9.5.78)$$

The form of the action functional for the relativistic medium is suggested by our previous experience with the relativistic clock. In place of the mass m in Eq. (9.4.2) we put the total rest energy $\int(\rho_0 + w_0)\, d^3\underset{\sim}{u}$. Thus

$$S \equiv -\int dt \int d^3\underset{\sim}{u}\, (\rho_0 + w_0)(-\dot{x}^2)^{\frac{1}{2}} . \qquad (9.5.79)$$

In varying this functional in order to obtain the dynamical equations it is important to remember that the metric tensor appearing in the quantity $\dot{x}^2 \equiv g_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu$ and elsewhere is an explicit function of the x^μ . Under an infinitesimal variation δx^μ in the functions $x^\mu(t, \underset{\sim}{u})$ one readily finds

$$\delta(-\dot{x}^2)^{\frac{1}{2}} = -(-\dot{x}^2)^{\frac{1}{2}} v^\mu v^\nu \delta x_{\mu\cdot\nu} , \qquad (9.5.80)$$

$$\delta v^\mu = v^\mu v^\nu v^\sigma \delta x_{\nu\cdot\sigma} + v^\nu \delta x^\mu{}_{,\nu} , \qquad (9.5.81)$$

$$\delta w_0 = - P^{\mu}_{\sigma} P^{\nu}{}_{\tau} x^{\sigma}{}_{,a} x^{\tau}{}_{,b} t^{ab} \delta x_{\mu\cdot\nu} \qquad (9.5.82)$$

and hence

$$\delta S \equiv \int T^{\mu\nu} \delta x_{\mu\cdot\nu} d^4x \quad , \qquad (9.5.83)$$

$$T^{\mu\nu} \equiv (\rho + w) v^{\mu} v^{\nu} + t^{\mu\nu} . \qquad (9.5.84)$$

The stationary action principle $\delta S = 0$ therefore leads to the dynamical equations

$$T^{\mu\nu}{}_{\cdot\nu} = 0 \quad , \qquad (9.5.85)$$

which, when combined with the mass conservation law (9.5.78), may be re-expressed in the form

$$v^{\mu}{}_{\cdot\nu} v^{\nu} = \rho^{-1} f^{\mu} \quad , \qquad (9.5.86)$$

$$f^{\mu} = - \bar{t}^{\mu\nu}{}_{\cdot\nu} \quad , \qquad (9.5.87)$$

$$\bar{t}^{\mu\nu} \equiv w v^{\mu} v^{\nu} + t^{\mu\nu} . \qquad (9.5.88)$$

The quantities f^{μ} and $\bar{t}^{\mu\nu}$ may be regarded as the world force density and the internal stress-energy density respectively. We note that the condition $v_{\mu} v^{\mu} = -1$ requires $v_{\mu} v^{\mu}{}_{\cdot\nu} = 0$ and hence

$$f^{\mu} v_{\mu} = 0 \quad , \qquad (9.5.89)$$

which also follows directly from Eqs. (9.5.87) and (9.5.88) together with the readily verified identity

$$(w v^{\mu})_{\cdot\mu} \equiv - t^{\mu\nu} v_{\mu\cdot\nu} . \qquad (9.5.90)$$

The quantity $T^{\mu\nu}$ is the total stress-energy density and is identical with the quantity defined by Eq. (9.5.52). This follows at once from the easily computed derivatives

$$\partial(-\dot{x}^2)/\partial g_{\mu\nu} \equiv -\frac{1}{2}(-\dot{x}^2)^{\frac{1}{2}} v^\mu v^\nu \quad , \tag{9.5.91}$$

$$\partial w_0/\partial g_{\mu\nu} \equiv -\frac{1}{2} P^\mu{}_\sigma P^\nu{}_\tau x^\sigma{}_{,a} x^\tau{}_{,b} t^{ab}. \tag{9.5.92}$$

We postpone discussion of the propagation of small disturbances in the relativistic elastic medium to the following section, in which the theory of Green's functions and Poisson brackets is developed for the more general system involving a dynamical gravitational field and a framework of clocks in interaction with the medium.

Appendix to Part I

(9.A) Green's functions for Fermi systems.

In addition to the real dynamical variables ϕ^i of Sections 2 and 3, we introduce Hermitian variables $\psi^{\mathbf{i}}$ satisfying Fermi statistics. In the semi-classical approximation the $\psi^{\mathbf{i}}$ commute with the ϕ^i but anticommute among themselves. The same conventions as in the text will be adopted for associating indices with the point labels x and z. Indices associated with the anti-commuting variables, however, will be identified by the use of boldface type.

Since the $\psi^{\mathbf{i}}$ anticommute they must be contained either linearly or in completely antisymmetric combinations in all dynamical quantities. Therefore boldface indices induced by repeated variational differentiation with respect to the $\psi^{\mathbf{i}}$ will anticommute among themselves while commuting with any light face indices induced by variational differentiation with respect to the ϕ^i. Such indices will be written in the order in which the variational differentiations are performed, and the type of derivative involved will always be the so-called "right" derivative. Thus the variation in a physical observable A due to variations $\delta\phi^i$, $\delta\psi^{\mathbf{i}}$ in the dynamical variables is given by

$$\delta A = \int (A_{,i}\,\delta\phi^i + A_{,\mathbf{i}}\,\delta\psi^{\mathbf{i}})\,d^4x = \int (\delta\phi^i A_{,i} - \delta\psi^{\mathbf{i}} A_{,\mathbf{i}})\,d^4x \ , \qquad \text{(A.1)}$$

the variations $\delta\phi^i$, $\delta\psi^{\mathbf{i}}$ being assumed to have the same commutation properties as the dynamical variables themselves.

In the present case a dynamical quantity may be a group invariant without being a physical observable. A physical observable A (and also the action functional S) must not only be a group invariant but must also be composed out of combinations of the $\psi^{\mathbf{i}}$ of even degree only. If it is also real (Hermitian), then its variational derivatives of order 1, 2, 5, 6, 9, 10, etc. with respect to the $\psi^{\mathbf{i}}$ will be imaginary (anti-Hermitian) while its variational derivatives of order 3, 4, 7, 8, 11, 12, etc. will be real (Hermitian). Furthermore, regardless of its reality, its variational derivatives of even order with respect to the $\psi^{\underset{\sim}{i}}$ will commute with everything while its variational derivatives of odd order will anticommute among themselves. Reflecting this latter rule the formalism below will be set up in such a way that quantities bearing an even number of boldface indices will always commute with everything while those bearing an odd number will always anticommute among themselves. The symmetry and reality properties of the two kinds of indices will not, however, generally follow the pattern which holds for covariant differentiation.

In the following we shall simply rewrite the pertinent equations of Sections 2 and 3 in the modified forms necessary to account for the new variables $\psi^{\underset{\sim}{i}}$, commenting upon them only when points of clarification are needed. There are now two sets of dynamical equations

$$S_{,i} = 0 \qquad \text{and} \qquad S_{,\underset{\sim}{i}} = 0 \,. \tag{A.2}$$

The infinitesimal group transformation law (9.2.2) becomes

$$\begin{pmatrix} \delta\phi^i \\ \delta\psi^{\underset{\sim}{i}} \end{pmatrix} = \int \begin{pmatrix} R^i{}_{L'} \\ R^{\underset{\sim}{i}}{}_{L'} \end{pmatrix} \delta\xi^{L'} \, d^4x' \,, \tag{A.3}$$

while the condition (9.2.3) is replaced by

$$\int \left[\begin{pmatrix} R^i{}_{A,j'} & R^i{}_{A,\underset{\sim}{j}'} \\ R^{\underset{\sim}{i}}{}_{A,j'} & R^{\underset{\sim}{i}}{}_{A,\underset{\sim}{j}'} \end{pmatrix} \begin{pmatrix} R^{j'}{}_{B'} \\ R^{\underset{\sim}{j}'}{}_{B'} \end{pmatrix} - \begin{pmatrix} R^i{}_{B',j'} & R^i{}_{B',\underset{\sim}{j}'} \\ R^{\underset{\sim}{i}}{}_{B',j} & R^{\underset{\sim}{i}}{}_{B',\underset{\sim}{j}'} \end{pmatrix} \begin{pmatrix} R^{j'}{}_{A} \\ R^{\underset{\sim}{j}'}{}_{A} \end{pmatrix} \right] d^4x'$$

$$= \int \begin{pmatrix} R^i{}_{L'} \\ R^{\underset{\sim}{i}}{}_{L'} \end{pmatrix} c^{L'}{}_{AB'} \, d^4x' \,. \tag{A.4}$$

Equation (9.2.4) remains unchanged. The quantities $R^i{}_A$, $R^{\underset{\sim}{i}}{}_A$, $R^i{}_{A,j'}$, $R^{\underset{\sim}{i}}{}_{A,j'}$, $R^{\underset{\sim}{i}}{}_{A,\underset{\sim}{j}'}$ are real while the quantity $R^i{}_{A,\underset{\sim}{j}'}$ is imaginary. The product of the latter with $R^{\underset{\sim}{j}'}{}_{B'}$ is, however, real, since it may be written in the form

$$R^i{}_{A,\underset{\sim}{j}'} R^{\underset{\sim}{j}'}{}_{B'} = \frac{1}{2} [R^i{}_{A,\underset{\sim}{j}'} , R^{\underset{\sim}{j}'}{}_{B'}] \ . \tag{A.5}$$

In these and all future equations the order of factors must be taken into account. It will be noted that the group parameters $\delta\xi^{L'}$ are here assumed to be real numbers which commute with everything.*

The condition (9.2.5) for group invariance becomes

$$\int (I_{,i} R^i{}_A + I_{,\underset{\sim}{i}} R^{\underset{\sim}{i}}{}_A)\, d^4x = 0 \ . \tag{A.6}$$

In virtue of the law

$$(X\,U)_{,\underset{\sim}{i}} = - X_{,\underset{\sim}{i}} U + XU_{,\underset{\sim}{i}} \ , \tag{A.7}$$

for functionals U of odd degree in the $\psi^{\underset{\sim}{i}}$, we obtain, on taking the variational derivative of Eq. (A.6) and remembering the symmetry properties for the induced indices,

$$\int \begin{pmatrix} I_{,ij'} & I_{,i\underset{\sim}{j}'} \\ I_{,\underset{\sim}{i}j'} & I_{,\underset{\sim}{i}\underset{\sim}{j}'} \end{pmatrix} \begin{pmatrix} R^{j'}{}_A \\ R^{\underset{\sim}{j}'}{}_A \end{pmatrix} d^4x' = - \int \begin{pmatrix} I_{,j'} R^{j'}{}_{A,i} + I_{,\underset{\sim}{j}'} R^{\underset{\sim}{j}'}{}_{A,i} \\ I_{,j'} R^{j'}{}_{A,\underset{\sim}{i}} + I_{,\underset{\sim}{j}'} R^{\underset{\sim}{j}'}{}_{A,\underset{\sim}{i}} \end{pmatrix} d^4x' , \tag{A.8}$$

which, when applied to the action functional S , yields the relation demonstrating the group invariance of the dynamical equations:

---

*The possibility of anticommuting group parameters also exists, but since it has no apparent physical interest we do not consider it. It is encountered, for example, in the gauge groups of massless fields having spins 3/2, 7/2, 9/2, etc. Since these groups are Abelian the structure constants vanish. The general non-Abelian case would involve anticommuting structure constants.

$$\begin{pmatrix} \delta S_{,i} \\ \delta S_{,\underset{\sim}{i}} \end{pmatrix} = -\int d^4x' \int d^4z \begin{pmatrix} R^{j'}{}_{A,i} & -R^{\underset{\sim}{j'}}{}_{A,i} \\ R^{j'}{}_{A,\underset{\sim}{i}} & R^{\underset{\sim}{j'}}{}_{A,\underset{\sim}{i}} \end{pmatrix} \begin{pmatrix} S_{,j'} \\ S_{,\underset{\sim}{j'}} \end{pmatrix} \delta\xi^A . \qquad \text{(A.9)}$$

The minus sign on $-R^{\underset{\sim}{j'}}{}_{A,i}$ results from an interchange of anticommuting factors.

When the dynamical equations are satisfied we have

$$\int \begin{pmatrix} S_{,ij'} & S_{,i\underset{\sim}{j'}} \\ S_{,\underset{\sim}{i}j'} & S_{,\underset{\sim}{i}\underset{\sim}{j'}} \end{pmatrix} \begin{pmatrix} R^{j'}{}_{A} \\ R^{\underset{\sim}{j'}}{}_{A} \end{pmatrix} d^4x' = 0 , \qquad \text{(A.10)}$$

which shows that the solutions of the equation

$$\int \begin{pmatrix} S_{,ij'} & S_{,i\underset{\sim}{j'}} \\ S_{,\underset{\sim}{i}j'} & S_{,\underset{\sim}{i}\underset{\sim}{j'}} \end{pmatrix} \begin{pmatrix} \delta_A^{\pm}\phi^{j'} \\ \delta_A^{\pm}\psi^{\underset{\sim}{j'}} \end{pmatrix} d^4x' = -\epsilon \begin{pmatrix} A_{,i} \\ A_{,\underset{\sim}{i}} \end{pmatrix} \qquad \text{(A.11)}$$

for the variations induced by the change $S \to S + \epsilon A$ in the action are not well defined but are determined only up to a group transformation (A.3). To render the solutions unique we introduce two sets of real functions $P_{\underset{\sim}{i}}{}^{A}$, $Q_{\underset{\sim}{i}A}$ and two sets of imaginary functions $P_i{}^A$, Q_{iA} such that the continuous matrices

$$\begin{pmatrix} F_{ij'} & F_{i\underset{\sim}{j'}} \\ F_{\underset{\sim}{i}j'} & F_{\underset{\sim}{i}\underset{\sim}{j'}} \end{pmatrix} \equiv \begin{pmatrix} S_{,ij'} & S_{,i\underset{\sim}{j'}} \\ S_{,\underset{\sim}{i}j'} & S_{,\underset{\sim}{i}\underset{\sim}{j'}} \end{pmatrix} + \int \begin{pmatrix} P_i{}^AQ_{j'A} & P_i{}^AQ_{\underset{\sim}{j'}A} \\ P_{\underset{\sim}{i}}{}^AQ_{j'A} & P_{\underset{\sim}{i}}{}^AQ_{\underset{\sim}{j'}A} \end{pmatrix} d^4z , \qquad \text{(A.12)}$$

$$F_{AB'} \equiv \int (Q_{iA}R^{i}{}_{B'} + Q_{\underset{\sim}{i}A}R^{\underset{\sim}{i}}{}_{B'})\, d^4x , \qquad \text{(A.13)}$$

$$F_A{}^{B'} \equiv \int (R^{i}{}_{A}P_i{}^{B'} - R^{\underset{\sim}{i}}{}_{A}P_{\underset{\sim}{i}}{}^{B'})\, d^4x , \qquad \text{(A.14)}$$

are all nonsingular wave operators. Use of the Green's functions for the first of these wave operators yields solutions of Eq. (A.11) satisfying the supplementary condition

$$\int (Q_{iA}\delta_A{}^{\pm}\phi^i + Q_{\underset{\sim}{i}A}\delta_A{}^{\pm}\psi^{\underset{\sim}{i}})\, d^4x = 0 . \tag{A.15}$$

We note that $F_{ij'}$, $F_{AB'}$, $F_A{}^{B'}$ are real while $F_{\underset{\sim}{i}j'}$, $F_{i\underset{\sim}{j}'}$, $F_{\underset{\sim}{i}\underset{\sim}{j}'}$ are imaginary. Correspondingly, $G^{\pm ij'}$, $G^{\pm i\underset{\sim}{j}'}$, $G^{\pm\underset{\sim}{i}j'}$, $G^{\pm AB'}$, $G^{\pm}{}_A{}^{B'}$ must be real while $G^{\pm\underset{\sim}{i}\underset{\sim}{j}'}$ is imaginary.

The defining equations for the Green's functions of the wave operator (A.12) are

$$\int \begin{pmatrix} F_{ik''} & F_{i\underset{\sim}{k}''} \\ F_{\underset{\sim}{i}k''} & F_{\underset{\sim}{i}\underset{\sim}{k}''} \end{pmatrix} \begin{pmatrix} G^{\pm k''j'} & G^{\pm k''\underset{\sim}{j}'} \\ G^{\pm\underset{\sim}{k}''j'} & G^{\pm\underset{\sim}{k}''\underset{\sim}{j}'} \end{pmatrix} d^4x'' = - \begin{pmatrix} \delta_i{}^{j'} & 0 \\ 0 & \delta_{\underset{\sim}{i}}{}^{\underset{\sim}{j}'} \end{pmatrix} , \tag{A.16a}$$

$$\int \begin{pmatrix} G^{\pm ik''} & G^{\pm i\underset{\sim}{k}''} \\ G^{\pm\underset{\sim}{i}k''} & G^{\pm\underset{\sim}{i}\underset{\sim}{k}''} \end{pmatrix} \begin{pmatrix} F_{k''j'} & F_{k''\underset{\sim}{j}'} \\ F_{\underset{\sim}{k}''j'} & F_{\underset{\sim}{k}''\underset{\sim}{j}'} \end{pmatrix} d^4x'' = - \begin{pmatrix} \delta_{j'}{}^{i} & 0 \\ 0 & \delta_{\underset{\sim}{j}'}{}^{\underset{\sim}{i}} \end{pmatrix} . \tag{A.16b}$$

We do not repeat the proof that one of these equations follows from the other. It carries through just as in the text in spite of the presence of anticommuting quantities, provided the factors are placed in the order indicated. Again a Huygens' principle can be set up for solutions $\delta\phi^i$, $\delta\psi^{\underset{\sim}{i}}$ of the homogeneous equation defined by the wave operator, namely

$$\begin{pmatrix} \delta\phi^i \\ \delta\psi^{\underset{\sim}{i}} \end{pmatrix} = \int_\Sigma d\Sigma_{\mu'} \int d^4x'' \int d^4z \begin{pmatrix} G^{ij''} & G^{i\underset{\sim}{j}''} \\ G^{\underset{\sim}{i}j''} & G^{\underset{\sim}{i}\underset{\sim}{j}''} \end{pmatrix} \begin{pmatrix} f^{\mu'}{}_{j''a} & f^{\mu'}{}_{j''\underset{\sim}{a}} \\ f^{\mu'}{}_{\underset{\sim}{j}''a} & f^{\mu'}{}_{\underset{\sim}{j}''\underset{\sim}{a}} \end{pmatrix} \begin{pmatrix} \delta\phi^a \\ \delta\psi^{\underset{\sim}{a}} \end{pmatrix} \tag{A.17}$$

where $G \equiv G^+ - G^-$ for all G's and where

$$\int \left[(\Phi_1{}^{i} \;\; \Phi_1{}^{\underset{\sim}{i}}) \begin{pmatrix} F_{ij'} & F_{i\underset{\sim}{j'}} \\ F_{\underset{\sim}{i}j'} & F_{\underset{\sim}{i}\underset{\sim}{j'}} \end{pmatrix} \begin{pmatrix} \Phi_2{}^{j'} \\ \Phi_2{}^{\underset{\sim}{j'}} \end{pmatrix} - (\Phi_1{}^{j'} \;\; \Phi_1{}^{\underset{\sim}{j'}}) \begin{pmatrix} F_{j'i} & F_{j'\underset{\sim}{i}} \\ F_{\underset{\sim}{j'}i} & F_{\underset{\sim}{j'}\underset{\sim}{i}} \end{pmatrix} \begin{pmatrix} \Phi_2{}^{i} \\ \Phi_2{}^{\underset{\sim}{i}} \end{pmatrix} \right] d^4x'$$

$$= \int d^4x' \int d^4x'' \frac{\partial}{\partial x^\mu} (\Phi_1{}^{i'} \;\; \Phi_1{}^{\underset{\sim}{i'}}) \begin{pmatrix} f^{\mu}{}_{\underset{\sim}{i'}j''} & f^{\mu}{}_{i'\underset{\sim}{j''}} \\ f^{\mu}{}_{i'j''} & f^{\mu}{}_{\underset{\sim}{i'}\underset{\sim}{j''}} \end{pmatrix} \begin{pmatrix} \Phi_2{}^{j''} \\ \Phi_2{}^{\underset{\sim}{j''}} \end{pmatrix} \qquad \text{(A.18)}$$

Here $f^{\mu}{}_{i'j''}$ is real while $f^{\mu}{}_{i'\underset{\sim}{j''}}$, $f^{\mu}{}_{\underset{\sim}{i'}j''}$, $f^{\mu}{}_{\underset{\sim}{i'}\underset{\sim}{j''}}$ are imaginary. The functions $\Phi_1{}^{i}$, $\Phi_1{}^{\underset{\sim}{i}}$, etc. may here be of either the commuting or anticommuting type.

Equations (9.3.17) through (9.3.31) of the text are replaced respectively by

$$\int \begin{pmatrix} F_{ij'} & F_{i\underset{\sim}{j'}} \\ F_{\underset{\sim}{i}j'} & F_{\underset{\sim}{i}\underset{\sim}{j'}} \end{pmatrix} \begin{pmatrix} \delta_A{}^{\pm}\phi^{j'} \\ \delta_A{}^{\pm}\psi^{\underset{\sim}{j'}} \end{pmatrix} d^4x' = -\epsilon \begin{pmatrix} A_{,i} \\ A_{,\underset{\sim}{i}} \end{pmatrix} , \qquad \text{(A.19)}$$

$$\begin{pmatrix} \delta_A{}^{\pm}\phi^{i} \\ \delta_A{}^{\pm}\psi^{\underset{\sim}{i}} \end{pmatrix} = -\epsilon \int \begin{pmatrix} G^{\pm ij'} & G^{\pm i\underset{\sim}{j'}} \\ G^{\pm \underset{\sim}{i}j'} & G^{\pm \underset{\sim}{i}\underset{\sim}{j'}} \end{pmatrix} \begin{pmatrix} A_{,j'} \\ A_{,\underset{\sim}{j'}} \end{pmatrix} d^4x' , \qquad \text{(A.20)}$$

$$\delta\xi^A = \int d^4x \int d^4z' \; G^{\pm AB'} (Q_{iB'}\delta_A{}^{\pm}\phi^{i} + Q_{\underset{\sim}{i}B'}\delta_A{}^{\pm}\psi^{\underset{\sim}{i}}) , \qquad \text{(A.21)}$$

$$\int (R^{j'}{}_A \;\; -R^{\underset{\sim}{j'}}{}_A) \begin{pmatrix} F_{j'i} & F_{j'\underset{\sim}{i}} \\ F_{\underset{\sim}{j'}i} & F_{\underset{\sim}{j'}\underset{\sim}{i}} \end{pmatrix} d^4x' = \int F_A{}^{B'} (Q_{iB'} \;\; Q_{\underset{\sim}{i}B'}) \, d^4z' , \qquad \text{(A.22)}$$

$$\int d^4x \int d^4z' \, F_A{}^{B'} (Q_{iB'} \;\; Q_{\underset{\sim}{i}B'}) \begin{pmatrix} G^{\pm ij'} & G^{\pm i\underset{\sim}{j'}} \\ G^{\pm \underset{\sim}{i}j'} & G^{\pm \underset{\sim}{i}\underset{\sim}{j'}} \end{pmatrix} = -(R^{j'}{}_A \;\; -R^{\underset{\sim}{j'}}{}_A) , \qquad \text{(A.23)}$$

$$
\int \begin{pmatrix} F_{ik''} & F_{i\underset{\sim}{k''}} \\ F_{\underset{\sim}{i}k''} & F_{\underset{\sim}{i}\underset{\sim}{k''}} \end{pmatrix} \left[\begin{pmatrix} G^{\pm k''j'} & G^{\pm k''\underset{\sim}{j'}} \\ G^{\pm \underset{\sim}{k''}j'} & G^{\pm \underset{\sim}{k''}\underset{\sim}{j'}} \end{pmatrix} - \begin{pmatrix} G^{\mp j'k''} & -G^{\mp \underset{\sim}{j'}k''} \\ -G^{\mp j'\underset{\sim}{k''}} & -G^{\mp \underset{\sim}{j'}\underset{\sim}{k''}} \end{pmatrix} \right] d^4x''
$$

$$
= -\int \left[\begin{pmatrix} F_{ik''} & F_{i\underset{\sim}{k''}} \\ F_{\underset{\sim}{i}k''} & F_{\underset{\sim}{i}\underset{\sim}{k''}} \end{pmatrix} - \begin{pmatrix} F_{k''i} & F_{\underset{\sim}{k''}i} \\ F_{k''\underset{\sim}{i}} & -F_{\underset{\sim}{k''}\underset{\sim}{i}} \end{pmatrix} \right] \begin{pmatrix} G^{\mp j'k''} & -G^{\mp \underset{\sim}{j'}k''} \\ -G^{\mp j'\underset{\sim}{k''}} & -G^{\mp \underset{\sim}{j'}\underset{\sim}{k''}} \end{pmatrix} d^4x''
$$

$$
= -\int d^4x'' \int d^4z \begin{pmatrix} P_i{}^A Q_{k''A} - P_{k''}{}^A Q_{iA} & P_I{}^A Q_{\underset{\sim}{k''}A} - P_{\underset{\sim}{k''}}{}^A Q_{iA} \\ P_{\underset{\sim}{i}}{}^A Q_{k''A} - P_{k''}{}^A Q_{\underset{\sim}{i}A} & P_{\underset{\sim}{i}}{}^A Q_{\underset{\sim}{k''}A} - P_{\underset{\sim}{k''}}{}^A Q_{\underset{\sim}{i}A} \end{pmatrix}
$$

$$
\times \begin{pmatrix} G^{\mp j'k''} & -G^{\mp \underset{\sim}{j'}k''} \\ -G^{\mp j'\underset{\sim}{k''}} & -G^{\mp \underset{\sim}{j'}\underset{\sim}{k''}} \end{pmatrix} , \qquad (A.31)
$$

$$\int d^4z' \int d^4z'' \, F_A{}^{B'} \, G^{\pm}_{B'}{}^{C''} \left(R^{j'}{}_{C''} \quad -R^{\underset{\sim}{j'}}{}_{C''}\right) = -\left(R^{j'}{}_A \quad R^{j'}_{\sim}{}_A\right), \tag{A.24}$$

$$\int \left(Q_{iA} \quad Q_{\underset{\sim}{i}A}\right) \begin{pmatrix} G^{\pm ij'} & G^{\pm i\underset{\sim}{j'}} \\ G^{\pm \underset{\sim}{i}j'} & G^{\pm \underset{\sim}{i}\underset{\sim}{j'}} \end{pmatrix} d^4x = \int G^{\pm}{}_A{}^{B'} \left(R^{j'}{}_{B'} \quad -R^{\underset{\sim}{j'}}{}_{B'}\right) \, d^4z' , \tag{A.25}$$

$$\int \left(Q_{iA}\delta_A{}^{\pm}\phi^i + Q_{\underset{\sim}{i}A}\delta_A{}^{\pm}\psi^{\underset{\sim}{i}}\right) d^4x$$

$$= \epsilon \int d^4x' \int d^4z' \, G^{\pm}{}_A{}^{B'} \left(A_{,j'} \, R^{j'}{}_{B'} + A_{,\underset{\sim}{j'}} \, R^{\underset{\sim}{j'}}{}_{B'}\right) , \tag{A.26}$$

$$\int \begin{pmatrix} F_{ij'} & F_{i\underset{\sim}{j'}} \\ F_{\underset{\sim}{i}j'} & F_{\underset{\sim}{i}\underset{\sim}{j'}} \end{pmatrix} \begin{pmatrix} R^{j'}{}_A \\ R^{\underset{\sim}{j'}}{}_A \end{pmatrix} d^4x' = \int \begin{pmatrix} P_i{}^{B'} \\ P_{\underset{\sim}{i}}{}^{B'} \end{pmatrix} F_{B'A} \, d^4z' , \tag{A.27}$$

$$\int d^4x' \int d^4z' \begin{pmatrix} G^{\pm ij'} & G^{\pm i\underset{\sim}{j'}} \\ G^{\pm \underset{\sim}{i}j'} & G^{\pm \underset{\sim}{i}\underset{\sim}{j'}} \end{pmatrix} \begin{pmatrix} P_{j'}{}^{B'} \\ P_{\underset{\sim}{j'}}{}^{B'} \end{pmatrix} F_{B'A} = -\begin{pmatrix} R^i{}_A \\ R^{\underset{\sim}{i}}{}_A \end{pmatrix}, \tag{A.28}$$

$$\int d^4z' \int d^4z'' \begin{pmatrix} R^i{}_{C''} \\ R^{\underset{\sim}{i}}{}_{C''} \end{pmatrix} G^{\pm C''B'} \, F_{B'A} = -\begin{pmatrix} R^i{}_A \\ R^{\underset{\sim}{i}}{}_A \end{pmatrix} , \tag{A.29}$$

$$\int \begin{pmatrix} G^{\pm ij'} & G^{\pm i\underset{\sim}{j'}} \\ G^{\pm \underset{\sim}{i}j'} & G^{\pm \underset{\sim}{i}\underset{\sim}{j'}} \end{pmatrix} \begin{pmatrix} P_{j'}{}^{B'} \\ P_{\underset{\sim}{j'}}{}^{B'} \end{pmatrix} d^4x' = \int \begin{pmatrix} R^i{}_A \\ R^{\underset{\sim}{i}}{}_A \end{pmatrix} G^{\pm AB'} \, d^4z , \tag{A.30}$$

$$
\begin{pmatrix} G^{\pm i j'} & G^{\pm i \underset{\sim}{j'}} \\ G^{\pm \underset{\sim}{i} j'} & G^{\pm \underset{\sim}{i}\underset{\sim}{j'}} \end{pmatrix} - \begin{pmatrix} G^{\mp j' i} & -G^{\mp \underset{\sim}{j'} i} \\ -G^{\mp j' \underset{\sim}{i}} & -G^{\mp \underset{\sim}{j'}\underset{\sim}{i}} \end{pmatrix}
$$

$$
= \int d^4x'' \int d^4z \int d^4z' \Bigg[G^{\pm B'A} \begin{pmatrix} R^{i}{}_{B'}Q_{k''A} & R^{i}{}_{B'}Q_{\underset{\sim}{k''}A} \\ R^{\underset{\sim}{i}}{}_{B'}Q_{k''A} & R^{\underset{\sim}{i}}{}_{B'}Q_{\underset{\sim}{k''}A} \end{pmatrix} \begin{pmatrix} G^{\mp j' k''} & -G^{\mp \underset{\sim}{j'} k''} \\ -G^{\mp j' \underset{\sim}{k''}} & -G^{\mp \underset{\sim}{j'}\underset{\sim}{k''}} \end{pmatrix}
$$

$$
- \begin{pmatrix} G^{\pm i k''} & G^{\pm i \underset{\sim}{k''}} \\ G^{\pm \underset{\sim}{i} k''} & G^{\pm \underset{\sim}{i}\underset{\sim}{k''}} \end{pmatrix} \begin{pmatrix} R^{j'}{}_{B'}Q_{k''A} & -R^{\underset{\sim}{j'}}{}_{B'}Q_{k'A} \\ R^{j'}{}_{B'}Q_{\underset{\sim}{k''}A} & R^{\underset{\sim}{j'}}{}_{B'}Q_{\underset{\sim}{k''}A} \end{pmatrix} G^{\mp B'A} \Bigg] ,
$$

(A.32)

$$
\delta_A {}^{\pm}B - \delta_B {}^{\mp}A
$$

$$
= \epsilon \int d^4x \int d^4x' \Bigg[(B_{,i} \quad B_{,\underset{\sim}{i}}) \begin{pmatrix} G^{\pm i j'} & G^{\pm i \underset{\sim}{j'}} \\ G^{\pm \underset{\sim}{i} j'} & G^{\pm \underset{\sim}{i}\underset{\sim}{j'}} \end{pmatrix} \begin{pmatrix} A_{,j'} \\ A_{,\underset{\sim}{j'}} \end{pmatrix}
$$

$$
- (A_{,j'} \quad A_{,\underset{\sim}{j'}}) \begin{pmatrix} G^{\mp j' i} & G^{\mp j' \underset{\sim}{i}} \\ G^{\mp \underset{\sim}{j'} i} & G^{\mp \underset{\sim}{j'}\underset{\sim}{i}} \end{pmatrix} \begin{pmatrix} B_{,i} \\ B_{,\underset{\sim}{i}} \end{pmatrix}
$$

$$
= \epsilon \int d^4x \int d^4x' \; (B_{,i} \quad B_{,\underset{\sim}{i}}) \left[\begin{pmatrix} G^{\pm i j'} & G^{\pm i \underset{\sim}{j'}} \\ G^{\pm \underset{\sim}{i} j'} & G^{\pm \underset{\sim}{i}\underset{\sim}{j'}} \end{pmatrix} - \begin{pmatrix} G^{\mp j' i} & -G^{\mp \underset{\sim}{j'} i} \\ -G^{\mp j' \underset{\sim}{i}} & -G^{\mp \underset{\sim}{j'}\underset{\sim}{i}} \end{pmatrix} \right] \begin{pmatrix} A_{,j'} \\ A_{,\underset{\sim}{j'}} \end{pmatrix}
$$

$$
= 0 \quad .
$$

(A.34)

Since the reciprocity theorem holds, a canonical transformation group may again be introduced in order to show that the Poisson brackets satisfy all the usual identities. We note that the Poisson bracket may be written in the form

$$(A, B) = \int d^4x \int d^4x' \; (A_{,i} \;\; A_{,\underset{\sim}{i}}) \begin{pmatrix} G^{ij'} & G^{i\underset{\sim}{j}'} \\ G^{\underset{\sim}{i}j'} & G^{\underset{\sim}{i}\underset{\sim}{j}'} \end{pmatrix} \begin{pmatrix} B_{,j'} \\ B_{,\underset{\sim}{j}'} \end{pmatrix} . \tag{A.35}$$

The functions $P_i{}^A$, $P_{\underset{\sim}{i}}{}^A$, Q_{iA}, $Q_{\underset{\sim}{i}A}$ may usually be chosen in practice in such a way that

$$F_{ij'} = F_{j'i}, \qquad F_{\underset{\sim}{i}j'} = F_{j'\underset{\sim}{i}}, \qquad F_{\underset{\sim}{i}\underset{\sim}{j}'} = -F_{\underset{\sim}{j}'\underset{\sim}{i}} . \tag{A.36}$$

In this circumstance

$$\left.\begin{aligned} G^{\pm ij'} &= G^{\mp j'i}, & G^{ij'} &= -G^{j'i}, \\ G^{\pm i\underset{\sim}{j}'} &= -G^{\mp \underset{\sim}{j}'i}, & G^{i\underset{\sim}{j}'} &= G^{\underset{\sim}{j}'i}, \\ G^{\pm \underset{\sim}{i}\underset{\sim}{j}'} &= -G^{\mp \underset{\sim}{j}'\underset{\sim}{i}}, & G^{\underset{\sim}{i}\underset{\sim}{j}'} &= G^{\underset{\sim}{j}'\underset{\sim}{i}}, \end{aligned}\right\} \tag{A.37}$$

and, in passing to the rigorous quantum theory, the commutator

$$i(A, B) \equiv [A, B] = \int (A_{,i} \;\; A_{,\underset{\sim}{i}}) \begin{pmatrix} [\phi^i, B] \\ [\psi^{\underset{\sim}{i}}, B] \end{pmatrix} d^4x$$

$$= \int d^4x \int d^4x' \; (A_{,i} \;\; A_{,\underset{\sim}{i}}) \begin{pmatrix} [\phi^i, \phi^{j'}] & -[\phi^i, \psi^{\underset{\sim}{j}'}] \\ [\psi^{\underset{\sim}{i}}, \phi^{j'}] & -\{\psi^{\underset{\sim}{i}}, \psi^{\underset{\sim}{j}'}\} \end{pmatrix} \begin{pmatrix} B_{,j'} \\ B_{,\underset{\sim}{j}'} \end{pmatrix} \tag{A.38}$$

may then be computed <u>as if</u> ϕ^i, $\psi^{\underset{\sim}{i}}$ satisfied the commutation relations*

---

* The curly brackets denote the anticommutator.

$$
\left.\begin{aligned}
[\phi^{i}, \phi^{j'}] &= i\, G^{ij'} \quad , \\
[\phi^{i}, \psi^{\boldsymbol{j}'}] &= -i\, G^{i\boldsymbol{j}'} \quad , \\
[\psi^{\boldsymbol{i}}, \phi^{j'}] &= i\, G^{\boldsymbol{i}j'} \quad , \\
\{\psi^{\boldsymbol{i}}, \psi^{\boldsymbol{j}'}\} &= -i\, G^{\boldsymbol{ij}'} \quad .
\end{aligned}\right\} \tag{A.39}
$$

Beyond this point we encounter problems of operator consistency which require a separate investigation. We note, however, that the propagation functions on the right of Eq. (A.39) possess the reality and symmetry properties demanded by the commutators or anticommutators on the left.

Part II

(9.6) The interaction of the gravitational field with a stiff elastic medium carrying a framework of clocks.

We have now reached the point in our discussion at which the gravitational field may be introduced as a dynamical entity. It may seem unusual that a study devoted to the quantization of the geometry of space-time should devote so much preliminary attention to the quantization of physical systems which occupy space-time. As has been pointed out in the Introduction, however, it is only by measurements performed with the aid of such systems that a meaning can be given to "space-time geometry" in the first place. Furthermore, the general theory of measurement (in which such systems play an essential role) and its mathematical analysis through the theory of Green's functions, which is basic to the covariant approach, had also to be developed first, in Sections 2 and 3.

We now put every thing together into a combined system. The constituent particles of the elastic medium will themselves be taken as relativistic clocks having rest masses m depending on action variables J. We may speak of an angle-and-action-variable field, Θ, J, which is, in effect, a conceptual idealization of an actual clock framework. The masses m may have an explicit dependence on $\underset{\sim}{u}$ as well as on J. The rest mass density will be given by

$$\rho_0 = n_0 m \;, \tag{9.6.1}$$

where n_0 is the particle number density in the Lagrangian system, which may itself depend on $\underset{\sim}{u}$. Conservation of particle number may be expressed in either of the forms

$$\dot{n}_0 = 0 \;, \qquad\qquad (nv^{\mu})_{\cdot\mu} = 0 \;. \tag{9.6.2}$$

Conservation of rest mass will follow in the present case as a consequence of the dynamical equations [see Eq. (9.6.14)].

Since the metric components $g_{\mu\nu}$ are now dynamical variables, subject to their own independent variations as functions of the space-time coordinates x^μ, it becomes important to make a clear distinction between the x^μ as point labels and the functions $x^\mu(t, \underset{\sim}{u})$ which describe the world lines of the constituent particles---a distinction which was unnecessary as long as the metric remained fixed, as in the preceding section. To assist in making this distinction we shall replace the symbols $x^\mu(t, \underset{\sim}{u})$ for the world-line functions by the symbols $z^\alpha(t, \underset{\sim}{u})$, $\alpha = 0, 1, 2, 3$. Furthermore, it will often be found convenient to regard the qunatities appearing in a given covariant expression sometimes as functions of the z's (and hence of t and $\underset{\sim}{u}$) and sometimes as functions of the x's. Tensor quantities regarded as functions of the z's will be written with Greek indices taken from the first part of the alphabet, while the same quantities regarded as functions of the x's will be written with Greek indices taken from the middle of the alphabet. The z- or x-dependence of quantities bearing no indices will generally be clear from the context. Since the z's, but not the x's, are dynamical variables the relation between the two types of quantities is not symmetric. Thus for a vector A^μ we have

$$A^\alpha = \delta^\alpha{}_\mu \int \delta(z, x)\, A^\mu\, d^4x \ , \tag{9.6.3}$$

$$A^\mu = \delta^\mu{}_\alpha \int dt \int d^3\underset{\sim}{u}\, \frac{\partial(z)}{\partial(t, \underset{\sim}{u})}\, \delta(x, z) A^\alpha \ , \tag{9.6.4}$$

from which it follows that dynamical variations in A^α and A^μ are related by

$$\delta A^\alpha = \delta^\alpha{}_\mu \int \delta(z, x)\, \delta A^\mu\, d^4x + A^\alpha{}_{,\beta}\, \delta z^\beta , \tag{9.6.5}$$

$$\delta A^\mu = \delta^\mu{}_\alpha \int dt \int d^3\underset{\sim}{u}\, \frac{\partial(z)}{\partial(t, \underset{\sim}{u})}\, \delta(x, z)\, \delta A^\alpha - A^\mu{}_{,\nu}\, \delta z^\nu . \tag{9.6.6}$$

It is also to be noted that although variation and differentiation with respect to x, t, or $\underset{\sim}{u}$ commute, variation and differentiation with respect to z do not. Thus we have

$$\begin{aligned} \delta(A^\alpha{}_{,\beta}) &= \delta(\dot{A}^\alpha t_{,\beta} + A^\alpha{}_{,a} u^a{}_{,\beta}) \\ &= (\delta A^\alpha)_{,\beta} - (\dot{A}^\alpha t_{,\gamma} + A^\alpha{}_{,a} u^a{}_{,\gamma})(\delta \dot{z}^\gamma t_{,\beta} + \delta z^\gamma{}_{,b} u^b{}_{,\beta}) \\ &= (\delta A^\alpha)_{,\beta} - A^\alpha{}_{,\gamma} \delta z^\gamma{}_{,\beta} . \end{aligned} \tag{9.6.7}$$

Equations (9.6.3) through (9.6.7) hold unchanged in form, except for the number of indices, for all tensors and tensor densities.

The action functional for the combined system may be taken in the form

$$S \equiv \int dt \int d^3\underset{\sim}{u}\, [n_0 J\, \dot{\Theta} - (n_0 m + w_0)(-\dot{z}^2)^{\frac{1}{2}}] - \int g^{\frac{1}{2}} R\, d^4x , \tag{9.6.8}$$

where R is the Riemann scalar. The dynamical equations are

$$0 = \frac{\delta S}{\delta J} \equiv n_0[\dot{\Theta} - \omega(-\dot{z}^2)^{\frac{1}{2}}] \equiv \frac{\partial(z)}{\partial(t, \underset{\sim}{u})}\, n(\Theta_{\cdot\alpha} v^\alpha - \omega), \tag{9.6.9}$$

$$0 = \frac{\delta S}{\delta \Theta} \equiv -n_0 \dot{J} \equiv -\frac{\partial(z)}{\partial(t, \underset{\sim}{u})}\, nJ_{\cdot\alpha} v^\alpha , \tag{9.6.10}$$

$$0 = \frac{\delta S}{\delta z^\alpha} \equiv \frac{\partial(z)}{\partial(t, \underset{\sim}{u})}\, T_\alpha{}^\beta{}_{\cdot\beta} , \tag{9.6.11}$$

$$0 = \frac{\delta S}{\delta g_{\mu\nu}} \equiv -G^{\mu\nu} + \tfrac{1}{2} T^{\mu\nu} , \tag{9.6.12}$$

where

$$G^{\mu\nu} \equiv g^{\frac{1}{2}} \, (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) . \qquad (9.6.13)$$

In the covariant forms of Eqs. (9.6.9) and (9.6.10) θ and J are to be understood as scalar fields. From the latter equation it follows that

$$\dot{m} = 0 \;, \qquad (9.6.14)$$

and hence the rest mass conservation law (9.5.78) holds.

The action (9.6.8) possesses two independent infinite dimensional invariance groups: the general coordinate transformation group and the group of transformations of the parameter t. Under the infinitesimal transformations $x'^{\mu} = x^{\mu} - \delta x^{\mu}$, $t' = t - \delta t$, the dynamical variables suffer the changes

$$\left.\begin{aligned} \delta J &= \dot{J}\,\delta t = 0 \;, \\ \delta\theta &= \dot{\theta}\,\delta t = \omega(-\dot{z}^2)^{\frac{1}{2}}\,\delta t \;, \\ \delta z^{\alpha} &= \dot{z}^{\alpha}\delta t - \delta x^{\alpha} \;, \\ \delta g_{\mu\nu} &= \delta x_{\mu\cdot\nu} + \delta x_{\nu\cdot\mu} \;, \end{aligned}\right\} \qquad (9.6.15)$$

which lead to the following characterization of an invariant A :

$$\omega \frac{\delta A}{\delta\theta} + v^{\alpha} \frac{\delta A}{\delta z^{\alpha}} = 0 \;, \qquad (9.6.16)$$

$$g^{\alpha\beta} \, \frac{\delta A}{\delta z^{\beta}} + 2 \frac{\partial(z)}{\partial(t, \underset{\sim}{u})} \left(\frac{\delta A}{\delta g_{\alpha\beta}}\right)_{\cdot\beta} = 0 \;. \qquad (9.6.17)$$

The quantity $\delta A / \delta g_{\alpha\beta}$ is here to be understood as obtained by first computing the variational derivative $\delta A/\delta g_{\mu\nu}$ and then transforming from the x's to the z's via Eq. (9.6.3).

In order to obtain the equations satisfied by small disturbances a number of somewhat cumbersome variational computations must first be carried out. An outline of the principal steps involved is given in Appendix B at the end of the chapter. The results, for variations $\delta^{\pm}J$, $\delta^{\pm}\Theta$, $\delta^{\pm}z^{\alpha}$, $\delta^{\pm}g_{\mu\nu}$ induced by the change $S \to S + \epsilon A$ in the action, are

$$\frac{\partial(z)}{\partial(t, \underset{\sim}{u})} n(\omega v^{\alpha}v^{\beta} s^{\pm}_{\alpha\beta} + v^{\alpha}\delta^{\pm}\Theta_{\cdot\alpha} - \frac{\partial\omega}{\partial J}\delta^{\pm}J) = -\epsilon\frac{\delta A}{\delta J}, \tag{9.6.18}$$

$$-\frac{\partial(z)}{\partial(t, \underset{\sim}{u})} n v^{\alpha} \delta^{\pm} J_{\cdot\alpha} = -\epsilon\frac{\delta A}{\delta\Theta}, \tag{9.6.19}$$

$$\begin{aligned}-\frac{\partial(z)}{\partial(t, \underset{\sim}{u})}\Big(T^{\beta\gamma}(2 s^{\pm}_{\alpha\beta\cdot\gamma} - s^{\pm}_{\beta\gamma\cdot\alpha}) + \{[(\rho+w) v_{\alpha}v^{\beta}v^{\gamma}v^{\delta} + 2v_{\alpha}v^{\gamma}t^{\beta\delta}\\ + 2v^{\beta}v^{\gamma}t_{\alpha}{}^{\delta} - v_{\alpha}v^{\beta}t^{\gamma\delta} - v^{\gamma}v^{\delta}t_{\alpha}{}^{\beta} - c_{\alpha}{}^{\beta\gamma\delta}] s^{\pm}_{\gamma\delta} + n\omega v_{\alpha}v^{\beta}\delta^{\pm}J\}_{\cdot\beta}\Big)\\ = -\frac{\delta A}{\delta z^{\alpha}},\end{aligned} \tag{9.6.20}$$

$$\begin{aligned}&\frac{1}{2} g^{\frac{1}{2}} (g^{\mu\sigma}g^{\nu\tau} - \frac{1}{2} g^{\mu\nu}g^{\sigma\tau}) g^{\rho\lambda}(\delta^{\pm}g_{\sigma\tau\cdot\rho\lambda} + \delta^{\pm}g_{\rho\lambda\cdot\sigma\tau} - \delta^{\pm}g_{\sigma\rho\cdot\tau\lambda} - \delta^{\pm}g_{\tau\lambda\cdot\sigma\rho})\\ &+ \frac{1}{2} g^{\frac{1}{2}} (g^{\mu\nu}R^{\sigma\tau} + g^{\sigma\tau}R^{\mu\nu} - 2 g^{\mu\sigma}R^{\nu\tau} - 2g^{\nu\sigma}R^{\mu\tau} + g^{\mu\sigma}g^{\nu\tau}R - \frac{1}{2} g^{\mu\nu}g^{\sigma\tau}R)\delta^{\pm}g_{\sigma\tau}\\ &- \frac{1}{2}(T^{\mu\nu}\delta^{\pm}z^{\sigma})_{\cdot\sigma} + \frac{1}{2} T^{\mu\sigma}\delta^{\pm}z^{\nu}{}_{\cdot\sigma} + \frac{1}{2} T^{\nu\sigma}\delta^{\pm}z^{\mu}{}_{\cdot\sigma} + \frac{1}{2} [(\rho+w) v^{\mu}v^{\nu}v^{\sigma}v^{\tau}\\ &+ 2 v^{\mu}v^{\sigma}t^{\nu\tau} + 2 v^{\nu}v^{\sigma}t^{\mu\tau} - v^{\mu}v^{\nu}t^{\sigma\tau} - v^{\sigma}v^{\tau}t^{\mu\nu} - c^{\mu\nu\sigma\tau}]s^{\pm}_{\sigma\tau}\\ &+ \frac{1}{2} n\omega v^{\mu}v^{\nu}\delta^{\pm}J = -\epsilon\frac{\delta A}{\delta g_{\mu\nu}},\end{aligned} \tag{9.6.21}$$

[18]

where $s^{+}_{\alpha\beta}$ and $s^{-}_{\alpha\beta}$ are the advanced and retarded forms of the invariant strain tensor,

$$s_{\alpha\beta} \equiv \frac{1}{2}(\delta z_{\alpha\cdot\beta} + \delta z_{\beta\cdot\alpha} + \delta g_{\alpha\beta}). \tag{9.6.22}$$

[The quantity, $\delta g_{\alpha\beta}$ is here related to the primary variation $\delta g_{\mu\nu}$ in the sense of Eq. (9.6.3), <u>not</u> Eq. (9.6.5).] This important tensor will be seen to play a rather ubiquitous role in the theory.[19] From Eqs. (9.6.15) it is apparent that variations produced by an infinitesimal coordinate transformation make no contribution to it. One may easily show that the strain tensor in the Lagrangian system [cf. Eqs. (9.5.37) and (9.5.38)] is obtained by projecting the invariant strain tensor into the local 3-spaces perpendicular to the world lines of the constituent particles:

$$s_{ab} \equiv \frac{1}{2}\delta\gamma_{ab} = P^{\alpha}{}_{\gamma}P^{\beta}{}_{\delta}z^{\gamma}{}_{,a}z^{\delta}{}_{,b}s_{\alpha\beta} \; . \qquad (9.6.23)$$

In order to solve the equations for the small disturbances it is necessary to impose supplementary conditions. The following conditions are convenient:

$$v^{\alpha}v^{\beta}s^{\pm}{}_{\alpha\beta} = 0 \; , \qquad (9.6.24)$$

$$(g^{\mu\sigma}g^{\nu\tau} - \frac{1}{2}g^{\mu\nu}g^{\sigma\tau})\delta^{\pm}g_{\sigma\tau\cdot\nu} = 0 \; . \qquad (9.6.25)$$

If these conditions are not already satisfied they can be imposed by first carrying out coordinate and parameter transformations (9.6.15) for which

$$\delta t = (-\dot{z}^2)^{-\frac{1}{2}}\int^{t}(-\dot{z}^2)^{\frac{1}{2}}v^{\alpha}v^{\beta}s^{\pm}{}_{\alpha\beta}\,dt \, , \qquad (9.6.26)$$

$$\delta x^{\mu} = \int \mathfrak{G}^{\pm\mu}{}_{\nu'}(g^{\nu'\tau'}g^{\sigma'\rho'} - \frac{1}{2}g^{\nu'\sigma'}g^{\tau'\rho'})\delta^{\pm}g_{\tau'\rho'\cdot\sigma'}\,d^4x \; , \qquad (9.6.27)$$

where the $\mathfrak{G}^{\pm\mu}{}_{\nu'}$ are the Green's functions satisfying the equation

$$g^{\frac{1}{2}}g^{\sigma\tau}\mathfrak{G}^{\pm\mu}{}_{\nu'\cdot\sigma\tau} - g^{\frac{1}{2}}R^{\mu}{}_{\sigma}\mathfrak{G}^{\pm\sigma}{}_{\nu'} = -\delta^{\mu}{}_{\nu'} \; . \qquad (9.6.28)$$

When the supplementary conditions hold, the equations for the small disturbances may be solved with the aid of Green's functions satisfying the equations

$$\int \begin{pmatrix} F_{JJ''} & F_{J\Theta''} & 0 & 0 \\ F_{\Theta J''} & 0 & 0 & 0 \\ F_{\alpha J''} & 0 & F_{\alpha\epsilon''} & F_{\alpha}{}^{\epsilon''\zeta''} \\ F^{\alpha\beta}{}_{J''} & 0 & F^{\alpha\beta}{}_{\epsilon''} & F^{\alpha\beta\epsilon''\zeta''} \end{pmatrix} \begin{pmatrix} G^{\pm}{}_{J''J'} & G^{\pm}{}_{J''\Theta'} & G^{\pm}{}_{J''}{}^{\gamma'} & G^{\pm}{}_{J''\gamma'\delta'} \\ G^{\pm}{}_{\Theta''J'} & G^{\pm}{}_{\Theta''\Theta'} & G^{\pm}{}_{\Theta''}{}^{\gamma'} & G^{\pm}{}_{\Theta''\gamma'\delta'} \\ G^{\pm\epsilon''}{}_{J'} & G^{\pm\epsilon''}{}_{\Theta'} & G^{\pm\epsilon''\gamma'} & G^{\pm\epsilon''}{}_{\gamma'\delta'} \\ G^{\pm}{}_{\epsilon''\zeta''J'} & G^{\pm}{}_{\epsilon''\zeta''\Theta'} & G^{\pm}{}_{\epsilon''\zeta''}{}^{\gamma'} & G^{\pm}{}_{\epsilon''\zeta''\gamma'\delta'} \end{pmatrix} d^4z''$$

$$= - \begin{pmatrix} \delta(z, z') & 0 & 0 & 0 \\ 0 & \delta(z, z') & 0 & 0 \\ 0 & 0 & \delta_{\alpha}{}^{\gamma'} & 0 \\ 0 & 0 & 0 & \delta^{\alpha\beta}{}_{\gamma'\delta'} \end{pmatrix}, \qquad (9.6.29)$$

where[20]

$$F_{JJ'} = - n(\partial\omega/\partial J)\, \delta(z, z') \,, \qquad (9.6.30)$$

$$F_{J\Theta'} = - F_{\Theta J'} = n\, v^{\alpha}\delta_{\cdot\alpha}(z, z') \,, \qquad (9.6.31)$$

$$F_{\alpha J'} = - [n\omega v_{\alpha} v^{\beta}\, \delta(z, z')]_{\cdot\beta} = - n\omega v^{\beta}[v_{\alpha}\delta(z, z')]_{\cdot\beta} \,, \qquad (9.6.32)$$

$$\begin{aligned} F_{\alpha\epsilon'} = {} & - T^{\gamma\delta}(\delta_{\alpha\epsilon'\cdot\gamma\delta} - R_{\gamma\alpha\delta}{}^{\zeta}\, \delta_{\zeta\epsilon'}) - [(v_{\alpha}v^{\gamma}t^{\beta\delta} + v_{\alpha}v^{\delta}t^{\beta\gamma} \\ & + v^{\beta}v^{\gamma}t_{\alpha}{}^{\delta} + v^{\beta}v^{\delta}t_{\alpha}{}^{\gamma} - v_{\alpha}v^{\beta}t^{\gamma\delta} - v^{\gamma}v^{\delta}t_{\alpha}{}^{\beta} \\ & - c_{\alpha}{}^{\beta\gamma\delta})\, \delta_{\gamma\epsilon'\cdot\delta}]_{\cdot\beta} \,, \end{aligned} \qquad (9.6.33)$$

$$F^{\epsilon'\zeta'} = -\frac{1}{2} T^{\beta\gamma}(2\delta_{\alpha\beta}{}^{\epsilon'\zeta'}{}_{\cdot\gamma} - \delta_{\beta\gamma}{}^{\epsilon'\zeta'}{}_{\cdot\alpha})$$

$$-\frac{1}{2} [(2 v_\alpha v^\gamma t^{\beta\delta} + 2 v^\beta v^\gamma t_\alpha{}^\delta - v_\alpha v^\beta t^{\gamma\delta} - v^\gamma v^\delta t_\alpha{}^\beta$$

$$- c_\alpha{}^{\beta\gamma\delta})\delta_{\gamma\delta}{}^{\epsilon'\zeta'}]_{\cdot\beta} \quad , \tag{9.6.34}$$

$$F^{\alpha\beta}{}_{J''} = \frac{1}{2} m v v^\alpha v^\beta \, \delta(z, z') \quad , \tag{9.6.35}$$

$$F^{\alpha\beta}{}_{\epsilon'} = -\frac{1}{2} (T^{\alpha\beta}\delta^\gamma{}_{\epsilon'})_{\cdot\gamma} + \frac{1}{2} T^{\alpha\gamma}\delta^\beta{}_{\epsilon'\cdot\gamma} + \frac{1}{2} T^{\beta\gamma}\delta^\alpha{}_{\epsilon'\cdot\gamma}$$

$$+\frac{1}{2} (v^\alpha v^\gamma t^{\beta\delta} + v^\alpha v^\delta t^{\beta\gamma} + v^\beta v^\gamma t^{\alpha\delta} + v^\beta v^\delta t^{\alpha\gamma} - v^\alpha v^\beta t^{\gamma\delta}$$

$$- v^\gamma v^\delta t^{\alpha\beta} - c^{\alpha\beta\gamma\delta}) \, \delta_{\gamma\epsilon'\cdot\delta} \quad , \tag{9.6.36}$$

$$F^{\alpha\beta\epsilon'\zeta'} = \frac{1}{2} g^{\frac{1}{2}}(g^{\alpha\gamma}g^{\beta\delta} - \frac{1}{2}g^{\alpha\beta}g^{\gamma\delta})g^{\eta\theta}\delta_{\gamma\delta}{}^{\epsilon'\zeta'}{}_{\cdot\eta\theta} - g^{\frac{1}{2}}R^{\alpha\gamma\beta\delta}\delta_{\gamma\delta}{}^{\epsilon'\zeta'}$$

$$+\frac{1}{2} g^{\frac{1}{2}}(g^{\alpha\beta}R^{\gamma\delta} + g^{\gamma\delta}R^{\alpha\beta} - g^{\alpha\gamma}R^{\beta\delta} - g^{\beta\gamma}R^{\alpha\delta} + g^{\alpha\gamma}g^{\beta\delta}R$$

$$-\frac{1}{2} g^{\alpha\beta}g^{\gamma\delta}R)\delta_{\gamma\delta}{}^{\epsilon'\zeta'} + \frac{1}{4} (2 v^\alpha v^\gamma t^{\beta\delta} + 2 v^\beta v^\gamma t^{\alpha\delta} - v^\alpha v^\beta t^{\gamma\delta}$$

$$- v^\gamma v^\delta t^{\alpha\beta} - c^{\alpha\beta\gamma\delta})\delta_{\gamma\delta}{}^{\epsilon'\zeta'} \quad , \tag{9.6.37}$$

and where

$$\delta_{\alpha\beta'} \equiv g_{\alpha\beta} \, \delta(z, z') \quad , \tag{9.6.38}$$

$$\delta_{\alpha\beta}{}^{\gamma'\delta'} \equiv \frac{1}{2} (\delta_\alpha{}^\gamma\delta_\beta{}^\delta + \delta_\alpha{}^\delta\delta_\beta{}^\gamma) \, \delta(z, z') \text{ , etc.} \tag{9.6.39}$$

Delta functions of z and z', rather than t, $\underset{\sim}{u}$ and t', $\underset{\sim}{u}'$, are used here in order to absorb the Jacobians which appear in Eqs. (9.6.18) (9.6.19), (9.6.20). The proof that the solutions obtained with the Green's functions of Eq. (9.6.29) actually satisfy the supplementary conditions (9.6.24), (9.6.25) is outlined in Appendix B.

It may be inferred immediately from Eq. (9.6.29) that the functions $G^{\pm}{}_{JJ'}$; $G^{\pm}{}_{J}{}^{\gamma'}$, $G^{\pm}{}_{J\gamma'\delta'}$, $G^{\pm}{}_{\Theta}{}^{\gamma'}$, $G^{\pm}{}_{\Theta\gamma'\delta'}$, $G^{\pm\epsilon}{}_{J'}$, $G^{\pm}{}_{\epsilon\zeta J'}$ vanish. The Poisson bracket of two invariants therefore takes the form

$$(A, B) = \int dt \int d^3\underset{\sim}{u} \int dt' \int d^3\underset{\sim}{u}' \left(\frac{\delta A}{\delta J}, \frac{\delta A}{\delta\Theta}, \frac{\delta A}{\delta z^{\alpha}}, \frac{\partial(z)}{\partial(t,\underset{\sim}{u})}\frac{\delta A}{\delta g_{\alpha\beta}}\right)$$

$$\times \begin{pmatrix} 0 & G_{J\Theta'} & 0 & 0 \\ G_{\Theta J'} & G_{\Theta\Theta'} & 0 & 0 \\ 0 & G^{\alpha}{}_{\Theta'} & G^{\alpha\gamma'} & G^{\alpha}{}_{\gamma'\delta'} \\ 0 & G_{\alpha\beta J'} & G_{\alpha\beta}{}^{\gamma'} & G_{\alpha\beta\gamma'\delta'} \end{pmatrix} \begin{pmatrix} \delta B/\delta J' \\ \delta B/\delta\Theta' \\ \delta B/\delta z^{\gamma'} \\ \frac{\partial(z')}{\partial(t',\underset{\sim}{u}')}\frac{\delta B}{\delta g_{\gamma'\delta'}} \end{pmatrix} \qquad (9.6.40)$$

where the G's are the propagation functions formed from the $G^{\pm}$'s .

The transition to the proper time, which is the first step in the process of passing completely to the intrinsic coordinate system provided by the elastic medium and its framework (or "field") of clocks, is carried out in just the same way as was done for the relativistic clock in Section 4 . Equation (9.4.43), with $(-\dot{x}^2)^{\frac{1}{2}}$ now replaced by $(-\dot{z}^2)^{\frac{1}{2}}$, is again used to define quantities taken with a definite numerical value of the intrinsic proper time $\omega^{-1}\Theta$. Equations (9.4.44) and (9.4.45) suffer only slight modification, now taking the forms

$$\partial f_{\tau}/\partial\tau = (v^{\alpha}f_{,\alpha})_{\tau} \quad , \qquad (9.6.41)$$

$$\delta^{\pm}f_{\tau} = (\delta^{\pm}f)_{\tau} + (v^{\alpha}f_{,\alpha})_{\tau}\,\omega^{-1}\left[-(\delta^{\pm}\Theta)_{\tau} + \tau(\partial\omega/\partial J)(\delta^{\pm}J)_{\tau}\right]$$
$$+ \epsilon\frac{\partial(\tau,\underset{\sim}{u})}{\partial(z_{\tau})} f_{\tau}(n\omega)^{-1}\left[\left(\frac{\delta A}{\delta J}\right)_{\tau} + \tau\frac{\partial\omega}{\partial J}\left(\frac{\delta A}{\delta\Theta}\right)_{\tau}\right] . \qquad (9.4.42)$$

The Poisson bracket of two invariants, A and B , regarded as explicit

It may be inferred immediately from Eq. (9.6.29) that the functions $G^{\pm}_{JJ'}$; $G^{\pm}{}_{J}{}^{\gamma'}$, $G^{\pm}_{J\gamma'\delta'}$, $G^{\pm}{}_{\Theta}{}^{\gamma'}$, $G^{\pm}_{\Theta\gamma'\delta'}$, $G^{\pm\epsilon}{}_{J'}$, $G^{\pm}_{\epsilon\zeta J'}$ vanish. The Poisson bracket of two invariants therefore takes the form

$$(A, B) = \int dt \int d^3\underset{\sim}{u} \int dt' \int d^3\underset{\sim}{u}' \left(\frac{\delta A}{\delta J}, \frac{\delta A}{\delta \Theta}, \frac{\delta A}{\delta z^{\alpha}}, \frac{\partial(z)}{\partial(t,\underset{\sim}{u})}\frac{\delta A}{\delta g_{\alpha\beta}}\right)$$

$$\times \begin{pmatrix} 0 & G_{J\Theta'} & 0 & 0 \\ G_{\Theta J'} & G_{\Theta\Theta'} & 0 & 0 \\ 0 & G^{\alpha}{}_{\Theta'} & G^{\alpha\gamma'} & G^{\alpha}{}_{\gamma'\delta'} \\ 0 & G_{\alpha\beta J'} & G_{\alpha\beta}{}^{\gamma'} & G_{\alpha\beta\gamma'\delta'} \end{pmatrix} \begin{pmatrix} \delta B/\delta J' \\ \delta B/\delta\Theta' \\ \delta B/\delta z^{\gamma'} \\ \frac{\partial(z')}{\partial(t',\underset{\sim}{u}')}\frac{\delta B}{\delta g_{\gamma'\delta'}} \end{pmatrix} \qquad (9.6.40)$$

where the G's are the propagation functions formed from the $G^{\pm}$'s .

The transition to the proper time, which is the first step in the process of passing completely to the intrinsic coordinate system provided by the elastic medium and its framework (or "field") of clocks, is carried out in just the same way as was done for the relativistic clock in Section 4 . Equation (9.4.43), with $(-\dot{x}^2)^{\frac{1}{2}}$ now replaced by $(-\dot{z}^2)^{\frac{1}{2}}$, is again used to define quantities taken with a definite numerical value of the intrinsic proper time $\omega^{-1}\Theta$. Equations (9.4.44) and (9.4.45) suffer only slight modification, now taking the forms

$$\partial f_{\tau}/\partial\tau = (v^{\alpha}f_{,\alpha})_{\tau} \quad , \qquad (9.6.41)$$

$$\delta^{\pm}f_{\tau} = (\delta^{\pm}f)_{\tau} + (v^{\alpha}f_{,\alpha})_{\tau}\,\omega^{-1}\left[-(\delta^{\pm}\Theta)_{\tau} + \tau(\partial\omega/\partial J)(\delta^{\pm}J)_{\tau}\right]$$

$$+ \epsilon\frac{\partial(\tau,\underset{\sim}{u})}{\partial(z_{\tau})} f_{\tau}(n\omega)^{-1}\left[\left(\frac{\delta A}{\delta J}\right)_{\tau} + \tau\frac{\partial\omega}{\partial J}\left(\frac{\delta A}{\delta\Theta}\right)_{\tau}\right] . \qquad (9.4.42)$$

The Poisson bracket of two invariants, A and B , regarded as explicit functionals of J_ζ $(z^\alpha)_\tau$, $g_{\mu\nu}$ becomes, after dropping the subscripts ,

$$(A,\ B) = \int d\tau \int d^3\underset{\sim}{u} \int d\tau' \int d^3\underset{\sim}{u}' \left(\frac{\delta A}{\delta J}, \frac{\delta A}{\delta z^\alpha}, \frac{\partial(z)}{\partial(\tau,\underset{\sim}{u})}\frac{\delta A}{\delta g_{\alpha\beta}}\right)$$

$$\times \begin{pmatrix} 0 & -G_{J\Theta'}v^{\gamma'}\omega^{-1} & 0 \\ -v^\alpha\omega^{-1}G_{\Theta J'} & \underset{\sim}{G}^{\alpha\gamma'} & G^\alpha{}_{\gamma'\delta'} \\ 0 & G_{\alpha\beta}{}^{\gamma'} - G_{\alpha\beta\Theta'}v^{\gamma'}\omega^{-1} & G_{\alpha\beta\gamma'\delta'} \end{pmatrix} \begin{pmatrix} \delta B/\delta J' \\ \delta B/\delta z^{\gamma'} \\ \dfrac{\partial(z')}{\partial(\tau',\underset{\sim}{u}')}\dfrac{\delta B}{\delta g_{\gamma'\delta'}} \end{pmatrix}, \tag{9.6.43}$$

where

$$\underset{\sim}{G}^{\alpha\gamma'} \equiv G^{\alpha\gamma'}_{\Theta} - G^\alpha{}_{\Theta'}v^{\gamma'}\omega^{-1} + v^\alpha\omega^{-1}G_{\Theta\Theta'}v^{\gamma'}\omega^{-1}$$
$$- v^\alpha\omega^{-1}G_{\Theta J'}v^{\gamma'}\omega^{-1}\tau'\frac{\partial\omega}{\partial J} - v^\alpha\omega^{-1}\tau\frac{\partial\omega}{\partial J}G_{J\Theta'}v^{\gamma'}\omega^{-1}. \tag{9.6.44}$$

The angle variables have now been dropped from the theory and the invariance condition (9.6.16) no longer plays a role. The condition (9.6.17), on the other hand, takes the modified form

$$g^{\alpha\beta}\frac{\delta A}{\delta z^\beta} + 2\frac{\partial(z)}{\partial(\tau,\underset{\sim}{\mu})}\left(\frac{\delta A}{\delta g_{\alpha\beta}}\right)_{\cdot\beta} = 0 . \tag{9.6.45}$$

From Eq. (9.6.29) it may be seen that the Green's functions $G^\pm{}_{J\Theta'}$, $G^\pm{}_{\Theta J'}$, $G^\pm{}_{\Theta\Theta'}$ satisfy the following equations, expressed in terms of the proper time:

$$\left.\begin{aligned} n_0\partial G^\pm{}_{\Theta J'}/\partial\tau &= -n_0\partial G^\pm{}_{J\Theta'}/\partial\tau = -\delta(\tau-\tau')\delta(\underset{\sim}{u},\underset{\sim}{u}') , \\ -n_0(\partial\omega/\partial J)\,G^\pm{}_{J\Theta'} &+ n_0\partial G^\pm{}_{\Theta\Theta'}/\partial\tau = 0 . \end{aligned}\right\} \tag{9.6.46}$$

The solutions of these equations are

$$\left.\begin{aligned} G^{\pm}_{\theta J'} &= - G^{\pm}_{J\theta'} = \pm n_0^{-1}\,\theta(\mp(\tau - \tau'))\,\delta(\underset{\sim}{u}, \underset{\sim}{u}'), \\ G^{\pm}_{\theta\theta'} &= \mp n_0^{-1}\,(\partial\omega/\partial J)\,\theta(\mp(\tau - \tau'))(\tau - \tau')\,\delta(\underset{\sim}{u}, \underset{\sim}{u}'), \end{aligned}\right\} \tag{9.6.47}$$

whence

$$\underset{\sim}{G}^{\alpha\gamma'} = G^{\alpha\gamma'} - G^{\alpha}{}_{\theta'}v^{\gamma'}\omega^{-1} \ . \tag{9.6.48}$$

It is shown in Appendix B that explicit expressions may also be obtained for the Green's functions $G^{\pm\alpha}{}_{\theta'}$, $G^{\pm}{}_{\alpha\beta\theta'}$. The results are

$$G^{\pm}{}_{\alpha\beta\theta'} = 0 \ , \tag{9.6.49}$$

$$G^{\pm\alpha}{}_{\theta'} = - v^{\alpha}\omega G^{\pm} \ , \tag{9.6.50}$$

where

$$G^{\pm}(z, z') \equiv \mp\,\theta(\mp(\tau - \tau'))\,\delta(\underset{\sim}{u}, \underset{\sim}{u}')\int_{\tau'}^{\tau} (\rho_0 + w_0)^{-1}\,d\tau \ , \tag{9.6.51}$$

$$- [(\rho + w)v^{\alpha}v^{\beta}G^{\pm}{}_{\cdot\beta}]_{\cdot\alpha} = - \delta\,(z, z') \ . \tag{9.6.52}$$

The Poisson bracket (9.6.43) may therefore be rewritten in the final form

$$(A, B) = \int d\tau \int d^3\underset{\sim}{u} \int d\tau' \int d^3\underset{\sim}{u}' \left(\frac{\delta A}{\delta J}\, , \frac{\delta A}{\delta z^{\alpha}}\, , \frac{\partial(z)}{\partial(\tau, \underset{\sim}{u})}\frac{\delta A}{\delta g_{\alpha\beta}}\right)$$

$$\times \begin{pmatrix} 0 & \delta(\underset{\sim}{u},\underset{\sim}{u}')v^{\gamma'}(n_0\omega)^{-1} & 0 \\ - v^{\alpha}(n_0\omega)^{-1}\delta(\underset{\sim}{u},\underset{\sim}{u}') & G^{\alpha\gamma'} + v^{\alpha}Gv^{\gamma'} & G^{\alpha}{}_{\gamma'\delta'} \\ 0 & G_{\alpha\beta}{}^{\gamma'} & G_{\alpha\beta\gamma'\delta'} \end{pmatrix} \begin{pmatrix} \delta B/\delta J' \\ \delta B/\delta z^{\gamma'} \\ \dfrac{\partial(z')}{\partial(\tau', \underset{\sim}{u}')}\dfrac{\delta B}{\delta g_{\gamma'\delta'}} \end{pmatrix} . \tag{9.6.53}$$

We call attention here to the fact that theories for special limiting cases may be obtained by stripping appropriate rows and columns from the matrix of this Poisson bracket, and by appropriately simplifying the defining equations for the Green's functions. For example, in the case of the pure gravitational field the Poisson bracket reduces to

$$(A, B) = \int d^4x \int d^4x' \frac{\delta A}{\delta g_{\mu\nu}} \tilde{G}_{\mu\nu\sigma'\tau'} \frac{\delta B}{\delta g_{\sigma'\tau'}} , \tag{9.6.54}$$

where the Green's functions satisfy the equation

$$\frac{1}{2} g^{\frac{1}{2}} (g^{\mu\sigma} g^{\nu\tau} - \frac{1}{2} g^{\mu\nu} g^{\sigma\tau}) g^{\rho\omega} G^{\pm}_{\sigma\tau\kappa'\lambda'\cdot\rho\omega}$$
$$- \frac{1}{2} g^{\frac{1}{2}} R^{\mu\sigma\nu\tau} G^{\pm}_{\sigma\tau\kappa'\lambda'} = - \delta^{\mu\nu}{}_{\kappa'\lambda'} \tag{9.6.55}$$

and the quantities A and B satisfy the invariance conditions

$$(\delta A/\delta g_{\mu\nu})_{\cdot\nu} = 0 , \qquad (\delta B/\delta g_{\mu\nu})_{\cdot\nu} = 0 . \tag{9.6.56}$$

Since the formalism for this case is considerable simpler than that for the general case, it may be argued that by introducing the elastic medium with its framework of clocks we are making the analysis unnecessarily complicated, at least insofar as the quantization of space-time geometry alone is concerned. It has been pointed out in the Introduction, however, that the theory of the pure gravitational field suffers from a major defect, namely the difficulty of finding <u>interesting</u> invariants, A and B, within its framework. Although the simpler formalism will doubtless find application to certain problems, the more general formalism is essential for gaining insight into the physical nature of the gravitational field and, in particular, for the analysis of the measurability problem.

Another example, that of the stressless medium in a fixed metric, can also be treated as a special case. The Green's functions for this example happen to be expressible in terms of rather simple geometrical structures, and since the solution has some points of interest a brief account of it is given in Appendix C.

We now consider the final step which must be taken in order to pass completely to the intrinsic coordinate system defined by the local proper time τ and the labels u^a. Since the quantities appearing in the Poisson bracket (9.6.53) are already regarded as functions of τ and $\underset{\sim}{u}$, our goal is practically in sight. All that remains is to introduce a basic set of invariants in terms of which all physically meaningful quantities for the combined system may be expressed. This can be accomplished by taking any completely descriptive set of tensor quantities for the system and projecting them onto the intrinsic coordinates with the aid of the derivatives $z^\alpha{}_{,a}$ and $\dot{z}^\alpha$ which then disappear from the theory. The simplest such set is composed of just the metric tensor itself together with the action variable field J.

In the intrinsic coordinate system the metric tensor becomes

$$g_{(0)(0)} \equiv \dot{z}^\alpha \dot{z}^\beta g_{\alpha\beta} = v^\alpha v_\alpha = -1 \quad , \tag{9.6.57}$$

$$g_{(0)a} \equiv \dot{z}^\alpha z^\beta{}_{,a} g_{\alpha\beta} = v_\beta z^\beta{}_{,a} \equiv v_a \quad , \tag{9.6.58}$$

$$g_{ab} \equiv z^\alpha{}_{,a} z^\beta{}_{,b} g_{\alpha\beta} = \gamma_{ab} - v_a v_b \quad , \tag{9.6.59}$$

of which the contravariant form is

$$g^{(0)(0)} = -1 + v_a v^a \quad , \tag{9.6.60}$$

$$g^{(0)a} = v^a \quad , \tag{9.6.61}$$

$$g^{ab} = \gamma^{ab} \quad , \tag{9.6.62}$$

where

$$\gamma_{ac}\gamma^{cb} = \delta_a{}^b \ , \qquad v^a \equiv \gamma^{ab}v_b \ . \tag{9.6.63}$$

Once the composition of the medium is given, through specification of the explicit forms of the functions $n_0(\underset{\sim}{u})$, $w_0(\underset{\sim}{u}, \gamma_{ab})$, $m(\underset{\sim}{u}, J)$, the ten quantities γ_{ab}, v_a, J completely determine the dynamical state of the gravitational field together with the medium and its field of clocks, insofar as this state has observational meaning relative to the system itself.

Poisson brackets for the components of the intrinsic metric may be obtained by first computing its variations. One readily varifies that

$$\delta g_{(0)(0)} = 2\, v^\alpha v^\beta s_{\alpha\beta} = 0 \ , \tag{9.6.64}$$

$$\delta g_{(0)a} = 2\, v^\alpha z^\beta{}_{,a} s_{\alpha\beta} \ , \tag{9.6.65}$$

$$\delta g_{ab} = 2\, z^\alpha{}_{,a} z^\beta{}_{,b} s_{\alpha\beta} \ . \tag{9.6.66}$$

Therefore, using the dot followed by a subscript (0) , a , or b, etc. to denote covariant differentiation with respect to an intrinsic coordinate (and with respect to the intrinsic 4-metric), we have, from Eq. (9.6.53),

$$\begin{aligned}(g_{ab}, g_{c'd'}) = {} & G_{abc'd'} + G_{abc'\cdot d'} + G_{abd'\cdot c'} + G_{ac'd'\cdot b} + G_{bc'd'\cdot a} \\ & + (G_{ac'} + v_a G v_{c'})_{\cdot bd'} + (G_{ad'} + v_a G v_{d'})_{\cdot bd'} \\ & + (G_{bc'} + v_b G v_{c'})_{\cdot ad'} + (G_{bd'} + v_b G v_{d'})_{\cdot ac'} \ , \end{aligned} \tag{9.6.67}$$

$$(g_{ab}, g_{(0')c'}) = G_{ab(0')c'} + G_{ab(0')\cdot c'} + G_{abc'\cdot(0')} + G_{a(0')c'\cdot b}$$
$$+ G_{b(0')c'\cdot a} + (G_{a(0')} - v_a G)_{\cdot bc'} + (G_{ac'} + v_a G v_{c'})_{\cdot b(0')}$$
$$+ (G_{b(0')} - v_b G)_{\cdot ac'} + (G_{bc'} + v_b G v_{c'})_{\cdot a(0')} \ , \qquad (9.6.68)$$

$$(g_{(0)a}, g_{(0')b'}) = G_{(0)a(0')b'} + G_{(0)a(0')\cdot b'} + G_{(0)ab'\cdot(0')} + G_{(0)(0')b'\cdot a}$$
$$+ G_{a(0')b'\cdot(0)} + (G_{(0)(0')} + G)_{\cdot ab'} + (G_{(0)b'} - G v_{b'})_{\cdot a(0')}$$
$$+ (G_{a(0')} - v_a G)_{\cdot(0)b'} + (G_{ab'} + v_a G v_{b'})_{\cdot(0)(0')} \ , \qquad (9.6.69)$$

$$(g_{ab}, J') = - [v_a (n_0 \omega)^{-1} \delta(\underset{\sim}{u}, \underset{\sim}{u}')]_{\cdot b} - [v_b (n_0 \omega)^{-1} \delta(\underset{\sim}{u}, \underset{\sim}{u}')]_{\cdot a} \ , \qquad (9.6.70)$$

$$(g_{(0)a}, J') = [(n_0\omega)^{-1} \delta(\underset{\sim}{u}, \underset{\sim}{u}')]_{\cdot a} - v_{a\cdot(0)} (n_0 \omega)^{-1} \delta(\underset{\sim}{u}, \underset{\sim}{u}') \ , \qquad (9.6.71)$$

$$(J, J') = 0 \ . \qquad (9.6.72)$$

Since the propagation functions appearing in these Poisson brackets have always previously been defined in manifestly covariant terms, tney may in particular be computed directly in the intrinsic system. The whole theory is thereby rendered invariant and completely intrinsic.

It will be noted that Eq. (9.6.64), which was originally imposed as a supplementary condition, is automatically satisfied once the angle variables have been eliminated from the theory and the proper time has been introduced. It is, in fact, the condition which is necessary in order that the parameter τ remain the proper time under the variations δz^α, $\delta g_{\mu\nu}$. The constancy of $g_{(0)(0)}$, which it expresses, implies that $g_{(0)(0)}$ must have vanishing Poisson bracket with everything. This is readily verified with the aid of Eq. (B.21) and (B.22) of the Appendix together with the evident relation

$$\partial G^{\pm}/\partial\tau = \mathfrak{G}^{\pm} \ . \qquad (9.6.73)$$

We have

$$(g_{(0)(0)}, g_{a'b'}) = v^{\alpha}v^{\beta}z^{\gamma'}{}_{,a'}z^{\delta'}{}_{,b'}[G_{\alpha\beta\gamma'\delta'} + G_{\alpha\beta\gamma'\cdot\delta'} + G_{\alpha\beta\delta'\cdot\gamma'}$$

$$+ 2\, G_{\alpha\gamma'\delta'\cdot\beta} + 2\,(G_{\delta\gamma'} + v_{\alpha}Gv_{\gamma'})_{\cdot\beta\delta'}$$

$$+ 2\,(G_{\alpha\delta'} + v_{\alpha}Gv_{\delta'})_{\cdot\beta\gamma'}]$$

$$= 2\, z^{\gamma'}{}_{,a'}z^{\delta'}{}_{,b'}[v^{\alpha}v^{\beta}(G_{\alpha\gamma'\delta'\cdot\beta} + \tfrac{1}{2}\, G_{\alpha B\gamma'\delta'})$$

$$+ v^{\alpha}v^{\beta}(G_{\alpha\gamma'\cdot\beta} + \tfrac{1}{2}\, G_{\alpha\beta\cdot\gamma'})_{\cdot\delta'} + v^{\alpha}v^{\beta}(G_{\alpha\delta'\cdot\beta}$$

$$+ \tfrac{1}{2}\, G_{\alpha\beta\delta'})_{\cdot\gamma'} - v^{\beta}(Gv_{\gamma'})_{\cdot\beta\delta'} - v^{\beta}(Gv_{\delta'})_{\cdot\beta\gamma'}]$$

$$= 2\, z^{\gamma'}{}_{,a'}z^{\delta'}{}_{,b'}[(\dot{G}v_{\gamma'})_{\cdot\delta'} + (\dot{G}v_{\delta'})_{\cdot\gamma'}$$

$$- \partial(Gv_{\gamma'})_{\cdot\delta'}/\partial\tau - \partial(Gv_{\delta'})_{\cdot\gamma'}/\partial\tau]$$

$$= 0\ , \qquad (9.6.74)$$

and, similarly,

$$(g_{(0)(0)}, g_{(0')a'}) = 0\ , \quad (g_{(0)(0)}, g_{(0')(0')}) = 0\ . \qquad (9.6.75)$$

Also, because of the τ independence of n_0 and ω ,

$$(g_{(0)(0)}, J') = 2[(n_0\omega)^{-1}\delta(\underset{\sim}{u}, \underset{\sim}{u}')]_{\cdot(0)} = 0. \qquad (9.6.76)$$

The dynamical behavior of the system in the intrinsic coordinate frame is entirely determined by the ten Einstein equations:

$$\left.\begin{aligned} G^{(0)(0)} &= -\frac{1}{2}T^{(0)(0)} , \\ G^{(0)a} &= -\frac{1}{2}T^{(0)a} , \\ G^{ab} &= -\frac{1}{2}T^{ab} . \end{aligned}\right\} \qquad (9.6.77)$$

The Einstein tensor-density on the left is a function of the γ_{ab} , v_a and their $\underset{\sim}{u}$ and τ derivatives. The stress-energy density of the medium, on the right, however, is a function only of undifferentiated variables. Explicitly,

$$\left.\begin{aligned} T^{(0)(0)} &= \rho_0 + w_0 + v_a v_b t^{ab} , \\ T^{(0)a} &= v_b t^{ab} , \\ T^{ab} &= t^{ab} , \end{aligned}\right\} \qquad (9.6.78)$$

or, in covariant form,

$$\left.\begin{aligned} T_{(0)(0)} &= \rho_0 + w_0 , \\ T_{(0)a} &= -(\rho_0 + w_0)v_a , \\ T_{ab} &= (\rho_0 + w_0)v_a v_b + t_{ab} ; \end{aligned}\right\} \qquad (9.6.79)$$

where

$$t_{ab} \equiv \gamma_{ac}\gamma_{bd}t^{bd} . \qquad (9.6.80)$$

From these facts we may draw the usual conclusions about the number of degrees of freedom possessed by the gravitational field. The elastic medium possesses four degrees of freedom at each point $\underset{\sim}{u}$, three of which are external vibrational ones while the fourth is the internal one associated with the clock. These degrees of freedom are dynamically described by the momentum-energy conservation law, which here becomes a necessary consequence of the four contracted Bianchi identities. This law, however, is only of the first differential order in the variables γ_{ab} , v_a , J. Moreover, four of these variables may be expressed in terms of first τ-derivatives of the other six through the so-called <u>initial value equations</u> (Foures-Bruhat,1956)

$$- 2 G_{(0)}{}^{(0)} = T_{(0)}{}^{(0)} = - (\rho_0 + w_0) \quad , \tag{9.6.81}$$

$$- 2 G_a{}^{(0)} = T_a{}^{(0)} = (\rho_0 + w_0)v_a + t_{ab}v^b \quad . \tag{9.6.82}$$

Therefore there exist eight independent combinations of the γ_{ab} , v_a , J which are associated with the four degrees of freedom of the medium. The remaining two independent combinations correspond to two independent combinations of second-order Einstein equations. These are associated with the two degrees of freedom (per point $\underset{\sim}{u}$) possessed by the gravitational field.

It should be remarked, of course, that the actual combinations of variables associated with the various degrees of freedom are exceedingly difficult to find. Although a great amount of effort has been directed toward their discovery---they being the so-called <u>canonical variables</u> of the Hamiltonian approach to general relativity---success has so far been limited to the case in which flatness conditiona are imposed at infinity (Arnowitt, Deser and Misner, 1960). On the other hand, it is clear that knowledge of these variables is not essential to the quantization program, nor even to the asking and answering of questions of fundamental physical importance. This is particularly well illustrated by the discussion of measurability contained in Section 8.

It was pointed out in Section 3 that in order to go beyond the semi-classical approximation the problems of factor-ordering and self-consistency of the operator Green's functions would have to be solved. A suggested mode of attack on these problems involved taking the commutators of a complete set of invariant dynamical variables with the dynamical equations written in some invariant form. In the present case the invariant dynamical equations would be those of (9.6.77), written with their factors in a definite order, and the invariant variables would be the γ_{ab}, v_a, J. On the other hand, it seems most undesirable to make the rigorous quantization of the gravitational field itself depend on the presence of another physical system, in spite of the fact that the fundamental geometrical nature of the gravitational field can be physically elucidated only through its effects on other systems. This becomes particularly obvious in the present case when one considers that the problem of rigorous quantization is intimately related to the problems of fluctuation phenomena and renormalization, which lie precisely in the domain for which the continuum description of the elastic medium breaks down. The phenomenological cut-off which had to be used in Section 5 to describe fluctuations in the medium could hardly be expected to fit easily with the complete absence of an a priori cut-off for the gravitational field. One hopes, therefore, that a consistency procedure can first be worked out for the pure gravitational Green's functions of Eq. (9.6.55), without the necessity of finding invariant dynamical equations, and then later be extended to the case in which other systems are present.

(9.7) The quantized gravitational field.

Although the quantum description of the gravitational field has many points of similarity to conventional quantum field theory, it nevertheless seems incapable---or capable only with difficulty---of incorporating certain conventionally accepted notions. Nowhere is this better illustrated than in the problem of defining an energy-density for the gravitational field or, in more technical terms, a generator of infinitesimal space-time displacements. If is not actually difficult to formulate the problem, or at any rate it is not difficult once an intrinsic coordinate system has been set up, as in the preceding section. A displacement $\delta\tau$, δu^a with respect to the intrinsic coordinates is described by a variation in the z^α of amount

$$\delta z^\alpha = \dot{z}^\alpha \delta\tau + \delta z^\alpha{}_{,a} \delta u^a \quad , \tag{9.7.1}$$

with no accompanying variation in $\mathcal{e}$, J, or $g_{\mu\nu}$. The change in the explicit form of the action (9.6.8) which generates this variation is given by

$$\begin{aligned} \delta S &\equiv -\int T^{\alpha\beta} \delta z_{\alpha\cdot\beta} \, d^4 z \\ &\equiv -\int d\tau \int d^3\underset{\sim}{u} \left(T_{(0)}{}^{(0)} \delta\tau_{\cdot(0)} + T_{(0)}{}^{a} \delta\tau_{\cdot a} + T_a{}^{(0)} \delta u^a{}_{\cdot(0)} + T_a{}^b \delta u^a{}_{\cdot b} \right). \end{aligned} \tag{9.7.2}$$

By the arguments at the end of Section 3 the variation which this change induces in an arbitrary local invariant Φ is expressible in the form

$$\delta\Phi = \left(\Phi, \int \left(T_{(0')}{}^{(0')} \delta\tau' + T_{a'}{}^{(0')} \delta u^{a'} \right) d^3\underset{\sim}{u}' \right)_{\tau'=\tau} + \Delta\Phi \; , \tag{9.7.3}$$

in which it is assumed that the clocks have been adjusted in such a way that the hypersurface τ = constant, through the point at which Φ is evaluated, is space-like. [21]

The difficulty which now appears, however, is that the extra term $\Delta\Phi$ will not generally vanish even when Φ is one of the basic variables γ_{ab}, v_a. This may be seen at once from the fact that both singly and doubly differentiated propagation functions occur in the Poisson Brackets (9.6.67) et al which are to be used in the computation of the first term of (9.7.3). In the process of passing from the Green's functions to the propagation functions, as in Eq. (9.3.58), the step function will get differentiated sufficiently so that extra terms will unavoidably appear.[22] Whether these extra terms can themselves be obtained through a simple process of taking a Poisson bracket with some appropriate quantity is unknown. The prospects for this, however, are not encouraging. The only case in which it has so far been found possible to introduce an energy concept for the gravitational field is that in which flatness conditions at infinity are assumed (Arnowitt, Deser and Misner, 1960). The total energy then acts as a "time" displacement generator for the canonical variables, but these variables are physically nonlocal and depend, themselves, on the asymptotic conditions.

The existence of a space-time displacement generator is, of course, not essential to the quantization program. For the rest of the chapter, in fact, we shall get along quite well without it. There still remains the question, however, of the most suitable variables with which to work in developing the theory further and, in particular, in developing useful approximation methods. From the point of view of the logical structure of general relativity the primary variables would seem to be the components of the metric tensor, since these are the quantities which give direct and immediate information on the geometry of space-time. In the intrinsic frame, however, the metric components describe dynamical properties of both the elastic medium and the gravitational field, simultaneously and in a highly interlocking fashion. The mutual interference of the two systems is very obvious in Eqs. (9.6.67) et al in which all the propagation functions $G_{abc'd'}$, $G_{abc'}$, $G_{ab'c'}$, $G_{ab'}$, G, $G_{a(\rho)b'c'}$, etc. appear at once. Moreover, a direct observation of the metric components would require measurements by means of additional instruments similar to the medium itself (with its clocks) and subject to the same disturbances. But the reason for introducing the medium in the first place was precisely to use it as the primary standard and to give it lasting utility by making it sufficiently stiff. What we seek here are not the components of the metric tensor but a set of variables which are better at describing the gravitational field itself independently of the medium, as well as a set of instruments for measuring these variables having properties supplementing those of the medium.

The propagation of the gravitational field itself is principally described by the functions $G_{\alpha\beta\gamma'\delta'}$. To be sure, these functions depend to a certain extent on the elastic medium, just as the functions $G_{\alpha\beta'}$ and G depend to a certain extent on the gravitational field. However, near the light cone the behavior of $G_{\alpha\beta\gamma'\delta'}$ is determined by the gravitational field alone.[23] Because of the occurrence of derivatives of propagation functions in the Poisson brackets (9.6.67), etc. it would at first sight appear to be very difficult to obtain even an approximate separation of the mathematical description of the system into a gravitational part and a part referring to the medium. However, closer inspection shows that these derivatives occur in just the right way to make such a separation possible, provided the field satisfies a weakness condition which we may leave somewhat vague for the moment but which will be made more precise presently.

This possibility stems from the fact that the terms involving the differentiated propagation functions in Eqs.(9.6.67) et al have their origin in the terms $\delta z_{\alpha\cdot\beta} + \delta z_{\beta\cdot\alpha}$ contained in the invariant strain tensor appearing in the variations (9.6.64),(9.6.65),(9.6.66). Consider now the Riemann tensor. A variation in the metric of amount $\delta z_{\alpha\cdot\beta} + \delta z_{\beta\cdot\alpha}$ is mathematically identical to a coordinate transformation and therefore produces a variation in the Riemann tensor given by

$$\delta R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta\cdot\epsilon}\,\delta z + R_{\epsilon\beta\gamma\delta}\,\delta z^{\epsilon}{}_{\cdot\alpha} + R_{\alpha\epsilon\gamma\delta}\,\delta z^{\epsilon}{}_{\cdot\beta} + R_{\alpha\beta\epsilon\delta}\,\delta z^{\epsilon}{}_{\cdot\gamma} + R_{\alpha\beta\gamma\epsilon}\,\delta z^{\epsilon}{}_{\cdot\delta} \quad . \tag{9.7.4}$$

It will be observed that the Riemann tensor occurs as a factor in every term of this variation. Therefore if we compute Poisson brackets of components of the Riemann tensor in the intrinsic coordinate system, every term which involves one of the propagation functions $G_{\alpha\beta\gamma'}$, $G_{\alpha\gamma'\delta'}$, $G_{\alpha\beta'}$, G will also involve the Riemann tensor as a factor. In states for which the Riemann tensor differs only slightly from zero these terms become negligible, and only the propagation functions $G_{\alpha\beta\gamma'\delta'}$ are left.

It therefore appears that variables suitable for describing the gravitational field by itself are simply the components of the Riemann tensor, whenever these components are small. That the Riemann tensor should thus enter so directly into the description of the gravitational field is, of course, not surprising. The presence or absence of a real gravitational field is, in fact, determined by the value of the Riemann tensor. If the Riemann tensor vanishes there is no gravitational field; if its components are small the gravitational field is "weak." Of course, certain components of the Riemann tensor may become large even for a "weak" field if the coordinate system is badly chosen. But this circumstance can be avoided with a good stiff medium by assigning the labels u^a and setting the clocks in an intelligent manner. The quantum fluctuations of the medium as well as oscillations induced by the "weak" gravitational field itself will then have negligible effect on the Riemann components. It is important to note that such oscillations produce only a relative (fractional) change in the components of the Riemann tensor whereas they produce an absolute change in the components of the metric tensor. This circumstance has its immediate reflection in the linearized theory of gravitation, in which the Riemann tensor is a group invariant[24] although the "potentials" themselves are not. It suggests, moreover, that the linearized theory should provide a good starting point for approximation procedures, and indeed it will do so provided it is not used to settle global questions or used indiscriminately in the ultra high energy domain where violent fluctuations occur and where the effect of the gravitational field on the light cone itself must be considered. In short, it must be used with caution, and the full rigorous theory must always be kept in mind.

The equations of the linearized theory, or what should more properly be called the "weak field approximation" when sources and hence nonlinearities are introduced, may be obtained from the results of the preceding sections by regarding both sources and fields as small disturbances in the vacuum. We begin with an empty flat space-time. Then, in given regions of interest, we introduce stiff elastic bodies of limited dimensions, each in a state of uniform rectilinear motion,[25] with oscillatory modes absent except for the zero point fluctuations, and each defining a local Minkowskian coordinate system with the aid of its own framework of synchronized clocks. The introduction of these bodies produces two results: (1) a change in the action functional for the system (previously a pure vanishing gravitational field), and (2) a deviation of space-time from flatness. Denoting the departure of the local metric from Minkowskian by $\delta g_{\mu\nu}$, we have, in the immediate vicinity of one of the elastic bodies,

$$(\eta^{\mu\sigma}\eta^{\nu\tau} - \tfrac{1}{2}\eta^{\mu\nu}\eta^{\sigma\tau})\eta^{\rho\lambda}\,(\delta g_{\sigma\tau,\rho\lambda} + \delta g_{\rho\lambda,\sigma\tau} - \delta g_{\sigma\rho,\tau\lambda} - \delta g_{\tau\lambda,\sigma\rho}) = -\,T^{\mu\nu}, \quad (9.7.5)$$

which is just Eq. (9.6.21) simplified for the present situation. Here we use Greek indices from the middle of the alphabet to refer to the intrinsic coordinate system defined by the body, instead of separating the equation into parts corresponding to the space and time indices a, b, etc., and (0). Similarly we shall replace the coordinate labels u^a and τ by x^μ, not forgetting, however, their intrinsic origin. The stress-energy density in this system is, of course, $(T^{\mu\nu}) = \mathrm{diag}\,(\rho_0, 0, 0, 0)$. We may ignore the dynamics of the clocks except when we come to consider measurements of time.

Next, we give to the gravitational field a "free" component in addition to that produced by the introduction of the elastic bodies, which may be described either semi-classically or in q-number terms, depending on the state in question. If the field is sufficiently weak the superposition principle will hold and this extra component may be lumped together with the $\delta g_{\mu\nu}$ of Eq. (9.7.5). It is to be understood that the superposition principle need hold only in the immediate vicinity of the elastic bodies where the intrinsic coordinate frames actually exist; it need not hold in the large. Thus the geometry of space-time may now depart widely from flatness over large distances. The condition for "weakness" of the field and local validity of the superposition principle becomes simply that the product of the Riemann tensor with the square of the linear dimensions of the elastic bodies shall be small compared with unity. This, in turn, imposes a limitation on the bodies themselves, namely, the ratios of their masses to their linear dimensions must be small compared to one.

The fact that space-time is now permitted to have an appreciable macroscopic curvature means that the word used to describe the bulk motion of the elastic bodies must be changed from "uniform" to "geodetic." Furthermore, the small scale curvature will induce internal oscillations. These oscillations are described by the homogeneous form of Eq. (9.6.20) which, in the present approximation takes the form

$$\rho_0 v^\sigma v^\tau (\delta z_{\mu,\sigma\tau} + \delta g_{\mu\sigma,\tau} - \tfrac{1}{2}\delta g_{\sigma\tau,\mu}) + t_\mu{}^\sigma{}_{,\sigma} = 0, \tag{9.7.6}$$

in which the clock variables have been neglected, the condition (9.6.64) has been used, and the invariant strain tensor and internal stress density are taken in the respective forms

$$s_{\mu\nu} = \frac{1}{2}(\delta z_{\mu,\nu} + \delta z_{\nu,\mu} + \delta g_{\mu\nu}), \tag{9.7.7}$$

$$t_\mu{}^\nu = -c_\mu{}^{\nu\sigma\tau} s_{\sigma\tau} \quad . \tag{9.7.8}$$

Multiplying Eq. (9.7.6) by ρ_0^{-1}, differentiating with respect to x^ν, symmetrizing the resulting expression in the indices μ and ν, and remembering that the derivatives of v^σ vanish (uniform original motion), one gets

$$v^\sigma v^\tau (s_{\mu\nu,\sigma\tau} - \delta R_{\mu\sigma\nu\tau}) + \frac{1}{2}(\rho_0{}^{-1} t_\mu{}^\sigma{}_{,\sigma})_{,\nu} + \frac{1}{2}(\rho_0{}^{-1} t_\nu{}^\sigma{}_{,\sigma})_{,\mu} = 0, \tag{9.7.9}$$

where, from Eqs.(B.7) and (B.8) of Appendix B,

$$\delta R_{\mu\sigma\nu\tau} \equiv \frac{1}{2}(\delta g_{\mu\nu;\sigma\tau} + \delta g_{\sigma\tau;\mu\nu} - \delta g_{\mu\tau;\sigma\nu} - \delta g_{\sigma\nu;\mu\tau}). \qquad (9.7.10)$$

Since the original space-time was assumed to be flat, expression (9.7.10) really represents the full Riemann tensor, and the symbol δ may be removed from in front of the $R_{\mu\sigma\nu\tau}$. Therefore, remembering the conditions (9.5.77), as well as the fact that $(v^{\mu}) = (1,0,0,0)$ in the intrinsic frame, we may write the spatial components of Eq.(9.7.9) in the form

$$\ddot{s}_{ab} = R_{a(0)b(0)} - \frac{1}{2}(\rho_0^{-1} t_{ac;c})_{,b} - \frac{1}{2}(\rho_0^{-1} t_{bc;c})_{,a}. \qquad (9.7.11)$$

This equation has been used by Weber (1960) as the basis of proposals for the direct experimental detection of gravitational waves. We shall also make important use of it in the analysis of the measurability of the gravitational field in the next section.

It will now be convenient to rewrite the basic equations of the weak field approximation in several alternative forms. In view of Eq. (9.7.10) equation (9.7.5) can be written

$$R^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}R = -\frac{1}{2}T^{\mu\nu}, \quad R_{\mu\nu} \equiv R_{\mu\sigma\nu}{}^{\sigma}, \quad R \equiv R_{\mu}{}^{\mu}, \qquad (9.7.12)$$

which is the linearized Einstein equation. It can also be written

$$R_{\mu\nu} = -\frac{1}{2}(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T), \qquad (9.7.13)$$

$$T \equiv T_{\mu}{}^{\mu}. \qquad (9.7.14)$$

From the linearized Bianchi identities

$$R_{\mu\nu\sigma\tau;\rho} + R_{\mu\nu\tau\rho;\sigma} + R_{\mu\nu\rho\sigma;\tau} \equiv 0, \qquad (9.7.15)$$

it follows that

$$(R^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}R)_{,\nu} \equiv 0 . \qquad (9.7.16)$$

which imposes the lowest order condition

$$T^{\mu\nu}{}_{,\nu} = 0 \qquad (9.7.17$$

on the stress-energy density. This is, of course, trivially satisfied for $(T^{\mu\nu}) = \mathrm{diag}\,(\rho_0, 0, 0, 0)$ in view of the (proper) time independence of ρ_0. The above equations are also applicable, however, to situations in which the elastic bodies are subjected to sudden impulses arising from internal devices which are introduced for the purposes of measuring relative velocities or rates of strain (as in the next section), provided the stress-energy of these additional devices is itself taken into account. By repeated use of the Bianchi identities as well as Eq. (9.7.13) one obtains the important equation[26]

$$\begin{aligned}\Box^2 R_{\mu\nu\sigma\tau} &\equiv R_{\mu\nu\sigma\tau,\rho}{}^{\rho} \\ &= -\frac{1}{2}\,(T_{\mu\sigma} - \frac{1}{2}\eta_{\mu\sigma}T)_{,\nu\tau} + (T_{\nu\tau} - \frac{1}{2}\eta_{\nu\tau}T)_{,\mu\sigma} \\ &\quad - (T_{\mu\tau} - \frac{1}{2}\eta_{\mu\tau}T)_{,\nu\sigma} - (T_{\nu\sigma} - \frac{1}{2}\eta_{\nu\sigma}T)_{,\mu\tau} \quad ,\end{aligned} \qquad (9.7.18)$$

which enables the source-associated components of the Riemann tensor to be obtained directly from the stress-energy distribution with the aid of the familiar relativistic Green's functions $D^{\pm}$ massless fields, which satisfy the equation

$$\Box^2 D^{\pm}(x - x') = -\delta(x - x') . \qquad (9.7.19)$$

The use of these Green's functions is, of course, valid only in local regions.

The components of the Riemann tensor are conveniently separated into two sets, analogous respectively to the electric and magnetic components of the electromagnetic tensor of Maxwell's theory:

$$E_{ab} \equiv R_{a(0)b(0)} , \tag{9.7.20}$$

$$H_{ab} \equiv \frac{1}{2} \epsilon_{acd} R_{cdb(0)} . \tag{9.7.21}$$

The 3-tensor E_{ab} is obviously symmetric. Its trace is found, with the aid of Eq. (9.7.13), to be

$$E_{aa} = - \frac{1}{4} (T_{(0)(0)} + T_{aa}). \tag{9.7.22}$$

H_{ab}, on the other hand, has vanishing trace but is not symmetric when moving masses are present. Its antisymmetric part is given by

$$\frac{1}{2} (H_{ab} - H_{ba}) = - \frac{1}{4} \epsilon_{abc} T_{(0)c} . \tag{9.7.23}$$

The algebraic identities satisfied by the Riemann tensor may be used to show that its components can be re-expressed in terms of E_{ab} and H_{ab} by the equations

$$\left.\begin{aligned} R_{a(0)b(0)} &= E_{ab} , \qquad R_{cdb(0)} = \epsilon_{cda} H_{ab} , \\ R_{abcd} &= - \epsilon_{abe} \epsilon_{cdf} E_{ef} + \frac{1}{2} \epsilon_{abe} \epsilon_{cdf} J_{ef} , \end{aligned}\right\} \tag{9.7.24}$$

$$J_{ab} \equiv \frac{1}{2} T_{ab} - \frac{1}{4} \delta_{ab} (T_{(0)(0)} + T_{cc}) . \tag{9.7.25}$$

Furthermore, using these equations together with the Bianchi identities, and introducing dyadic notation, one may show that $\underset{\sim}{E}$ and $\underset{\sim}{H}$ satisfy the following analogs of Maxwell's equations:

$$\left.\begin{aligned} \underset{\sim}{\nabla} \cdot \underset{\sim}{E} &= \underset{\sim}{\nabla} \cdot \underset{\sim}{J} , \\ \underset{\sim}{\nabla} \times \underset{\sim}{E} + \underset{\sim}{\dot{\tilde{H}}} &= 0 , \\ \underset{\sim}{\nabla} \cdot \underset{\sim}{H} &= 0 , \\ \underset{\sim}{\nabla} \times \underset{\sim}{\tilde{H}} - \underset{\sim}{\dot{E}} &= - \underset{\sim}{\ddot{J}} , \end{aligned}\right\} \qquad (9\text{-}7\text{-}26)$$

the tilde denoting the transposed dyadic. Finally, the second order equations (9-7-18) may be rewritten in the forms

$$\Box^2 E_{ab} = -\frac{1}{2}[\ddot{T}_{ab} - \frac{1}{2}\delta_{ab}\ddot{T} + (T_{(0)(0)} + \frac{1}{2}T)_{,ab} - \dot{T}_{a(0),b} - \dot{T}_{b(0),a}] , \qquad (9\text{-}7\text{-}27)$$

$$\Box^2 H_{ab} = -\frac{1}{2}\epsilon_{acd}[(\dot{T}_{bc} - \frac{1}{2}\delta_{bc}\dot{T})_{,d} - T_{c(0),db}] . \qquad (9\text{-}7\text{-}28)$$

It is of interest to note that Eqs. (9-7-11) and (9-7-27) agree with the Newtonian theory in the static limit and may, indeed, be used to fix the size of the units through comparison with experiment. In the static limit, with $T_{(0)(0)} = \rho_0$, Eq. (9-7-27) becomes

$$\nabla^2 E_{ab} = -\frac{1}{4}\rho_{0,ab} . \qquad (9\text{-}7\text{-}29)$$

In the Newtonian theory, on the other hand, we have a scalar potential ϕ satisfying

$$\nabla^2\phi = 4\pi G\rho_0 = \frac{1}{4}\rho_0 , \qquad (9\text{-}7\text{-}30)$$

in terms of which the equation of motion of the constituent particles of a medium may be expressed:

$$\delta\ddot{z}_{aa} = -\phi_{,a} - \rho_0^{-1} t_{ab,b} . \qquad (9\text{-}7\text{-}31)$$

Introducing the Newtonian strain tensor

$$s_{ab} = \frac{1}{2}(\delta z_{a,b} + \delta z_{b,a}) , \qquad (9\text{-}7\text{-}32)$$

we are therefore led to Eq. (9.7.11), provided we make the correspondence

$$E_{ab} \leftrightarrow -\phi_{,ab} \quad , \tag{9.7.33}$$

in agreement with Eqs. (9.7.29) and (9.7.30).

We may also note that only the 3-tensor E_{ab}, and not H_{ab}, has thus far appeared in the equations which determine the effect of the curvature of space-time on bodies which occupy space-time. This is because we have confined our attention to non-rotating bodies. It has been shown by Papapetrou (1951) and Pirani (1956) that when spin is introduced, the 3-tensor H_{ab} enters directly into the law of force. The dynamical equations which these authors give for a particle of mass m and spin angular momentum tensor $\Sigma^{\alpha\beta}$ satisfying

$$\Sigma^{\alpha\beta} = -\Sigma^{\beta\alpha} \quad , \qquad \Sigma_{\alpha\beta}\dot{z}^{\beta} = 0 \quad , \tag{9.7.34}$$

are

$$m\ddot{z}^{\alpha} = \dot{\Sigma}^{\alpha}{}_{\beta}\dot{z}^{\beta} + \frac{1}{2} R^{\alpha}{}_{\beta\gamma\delta}\dot{z}^{\beta}\Sigma^{\gamma\delta} \quad , \tag{9.7.35}$$

$$\dot{\Sigma}^{\alpha\beta} = (\Sigma^{\alpha}{}_{\gamma}\dot{z}^{\beta} - \Sigma^{\beta}{}_{\gamma}\dot{z}^{\alpha})\ddot{z}^{\gamma} \quad . \tag{9.7.36}$$

Here the dot denotes covariant proper time differentiation. That is, if the particle is imagined to belong to an ensemble defining a velocity field v^{α}, then

$$v^{\alpha} \equiv \dot{z}^{\alpha} \equiv \partial z^{\alpha}/\partial\tau \quad , \quad \ddot{z}^{\alpha} \equiv v^{\beta}v^{\alpha}{}_{\cdot\beta} \quad , \quad \dot{\Sigma}^{\alpha\beta} \equiv v^{\gamma}\Sigma^{\alpha\beta}{}_{\cdot\gamma} \text{ , etc.} \tag{9.7.37}$$

Equation (9.7.36) expresses the condition that the spin propagate along the world-line of the particle in as parallel a fashion as is consistent with its remaining purely spatial [as demanded by condition (9.7.34)] and therefore gives simply the general relativistic Thomas precession. Equation (9.7.36) yields a deviation from geodetic motion of the particle itself. The spin equation has been used by Schiff (1960) as the basis of proposals for measuring the general relativistic corrections to the ordinary special relativistic Thomas precession. The particle equation, on the other hand, will play a fundamental role in the next section in the analysis of the measurability of H_{ab}. In Appendix D it is shown how both equations are derivable from a single momentum-energy conservation law.

The commutation relations for the 3-tensors E_{ab} and H_{ab} can be obtained directly from the commutation relations for the components of the linearized Riemann tensor, which, since only the propagation functions $G_{\mu\nu\sigma'\tau'}$ are now involved, may be computed as if the $\delta g_{\mu\nu}$ satisfied the commutation relations

$$[\delta g_{\mu\nu}, \delta g_{\sigma'\tau'}] = i\, G_{\mu\nu\sigma'\tau'} \,. \tag{9.7.38}$$

In the weak field approximation the Green's functions from which these propagation functions are formed satisfy the equation [cf. Eq. (9.6.55)]

$$\frac{1}{2}(\eta^{\mu\sigma}\eta^{\nu\tau} - \frac{1}{2}\eta^{\mu\nu}\eta^{\sigma\tau})\Box^2 G^{\pm}_{\sigma\tau\rho'\lambda'} = -\delta^{\mu\nu}{}_{\rho'\lambda'} \,, \tag{9.7.39}$$

of which the solutions is

$$G^{\pm}_{\mu\nu\sigma'\tau'} = (\eta_{\mu\sigma}\eta_{\nu\tau} + \eta_{\mu\tau}\eta_{\nu\sigma} - \eta_{\mu\nu}\eta_{\sigma\tau})D^{\pm}(x' - x'). \tag{9.7.40}$$

By a straightforward computation, which makes use of Eqs. (9.7.10) and (9.7.19), one finds

$$[E_{ab}, E_{c'd'}] = [H_{ab}, H_{c'd'}]$$
$$= \frac{1}{4}(\delta^T{}_{ac}\delta^T{}_{bd} + \delta^T{}_{ad}\delta^T{}_{bc} - \delta^T{}_{ab}\delta^T{}_{cd})\nabla^4 D(x - x'), \tag{9.7.41}$$

$$[E_{ab}, H_{c'd'}] = -[H_{ab}, E_{c'd'}]$$

$$= \frac{1}{4} i \epsilon_{cef}(\delta^T{}_{ae}\delta^T{}_{bd} + \delta^T{}_{ad}\delta^T{}_{be} - \delta^T{}_{ab}\delta^T{}_{ed}) \nabla^2 D_{,f}(x - x'), \tag{9.7.42}$$

Where $\delta^T{}_{ab}$ is the transverse-field projection operator:

$$\delta^T{}_{ab} \equiv \delta_{ab} - \frac{\partial}{\partial x^a} \nabla^{-2} \frac{\partial}{\partial x^b} , \tag{9.7.43}$$

and where

$$D \equiv D^+ - D^- , \qquad \Box^2 D(x - x') = 0 . \tag{9.7.44}$$

Since the gravitational propagation functions $G_{\mu\nu\sigma'\tau'}$ are uncoupled to the matter propagation functions $G_{\mu\nu'}$ and G in the weak field approximation, the commutation relations (9.7.41) and (9.7.42) are also satisfied by the "free" components of E_{ab} and H_{ab} which remain after the retarded (or advanced) solutions of Eqs. (9.7.27) and (9.7.28) have been subtracted out. A study of these free components leads to the concept of gravitational quanta or gravitons. From the homogeneous forms of Eqs. (9.7.26) and the fact that the free components of E_{ab} and H_{ab} are symmetric and have vanishing trace, it is not at all difficult to see that the Fourier decompositions of E^{free}_{ab} and H^{free}_{ab} have the general forms

$$\underset{\sim}{E}^{free} = (4\pi)^{-\frac{3}{2}} \int k^2 [(\underset{\sim}{e}_1\underset{\sim}{e}_1 - \underset{\sim}{e}_2\underset{\sim}{e}_2)a_I + (\underset{\sim}{e}_1\underset{\sim}{e}_2 + \underset{\sim}{e}_2\underset{\sim}{e}_1)a_{II}] e^{ik_\mu x^\mu} (d^3\underset{\sim}{k}/\sqrt{k^0})$$

$$+ \text{h.c.}, \tag{9.7.45}$$

$$\underset{\sim}{H}^{free} = (4\pi)^{-\frac{3}{2}} \int k^2 [(\underset{\sim}{e}_1\underset{\sim}{e}_2 + \underset{\sim}{e}_2\underset{\sim}{e}_1)a_I - (\underset{\sim}{e}_1\underset{\sim}{e}_1 - \underset{\sim}{e}_2\underset{\sim}{e}_2)a_{II}] e^{ik_\mu x^u} (d^3\underset{\sim}{k}/\sqrt{k^0})$$

$$+ \text{h. c.} , \tag{9.7.46}$$

where $(k^\mu) - (\underset{\sim}{k}, k^0)$, $k^0 = k \equiv |\underset{\sim}{k}|$, and where $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ are the

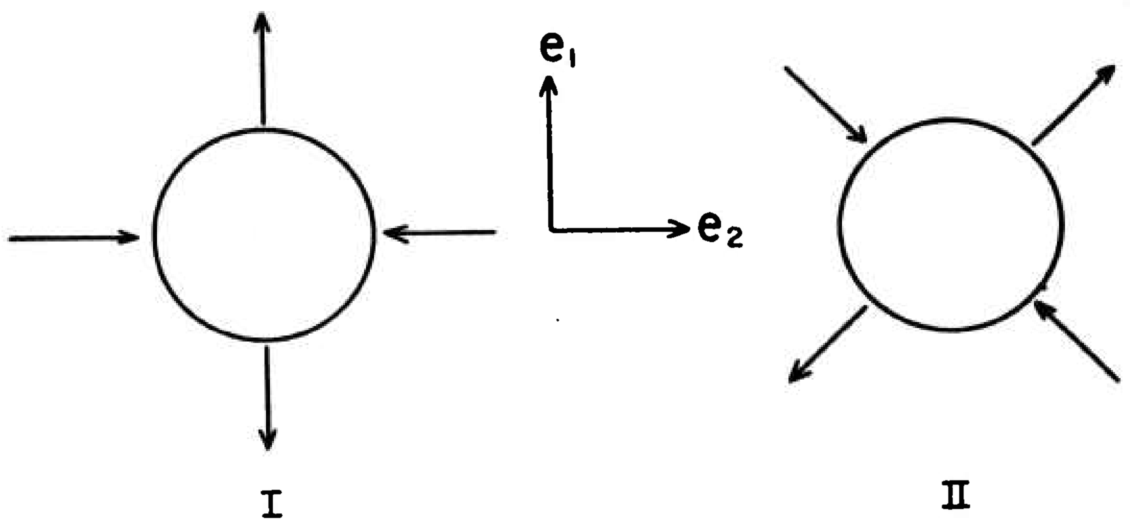

Fig. 9-2. Quadrupole strain induced in a spherical object by a plane gravitational wave propagating into the page, for the two polarization states, I and II .

mutually orthogonal unit vectors of Eq. (9.5.48) chosen, for definiteness, in a "right handed" fashion so that

$$\underset{\sim}{e}_i \times \underset{\sim}{e}_j = \epsilon_{ijk}\underset{\sim}{e}_k \ . \tag{9.7.47}$$

The "amplitudes" a_I, a_{II}, which are functions of $\underset{\sim}{k}$, define "polarization" states of their respective plane waves. The existence of two such states corresponds to the existence of two dynamical degrees of freedom per point $\underset{\sim}{u}$ (here $\underset{\sim}{x}$) which was mentioned at the end of Section 6. From the fact that

$$\left.\begin{aligned} \underset{\sim}{e}'_1\underset{\sim}{e}'_1 - \underset{\sim}{e}'_2\underset{\sim}{e}'_2 &= \underset{\sim}{e}_1\underset{\sim}{e}_2 + \underset{\sim}{e}_2\underset{\sim}{e}_1 = \underset{\sim}{e}_3 \times (\underset{\sim}{e}_1\underset{\sim}{e}_1 - \underset{\sim}{e}_2\underset{\sim}{e}_2) \ , \\ \underset{\sim}{e}'_1\underset{\sim}{e}'_2 + \underset{\sim}{e}'_2\underset{\sim}{e}'_1 &= -(\underset{\sim}{e}_1\underset{\sim}{e}_1 - \underset{\sim}{e}_2\underset{\sim}{e}_2) = \underset{\sim}{e}_3 \times (\underset{\sim}{e}_1\underset{\sim}{e}_2 + \underset{\sim}{e}_2\underset{\sim}{e}_1) , \end{aligned}\right\} \tag{9.7.48}$$

where

$$\underset{\sim}{e}'_1 \equiv 2^{-\frac{1}{2}}(\underset{\sim}{e}_1 + \underset{\sim}{e}_2) \ , \quad \underset{\sim}{e}'_2 \equiv 2^{-\frac{1}{2}}(-\underset{\sim}{e}_1 + \underset{\sim}{e}_2) \ , \tag{9.7.49}$$

it follows that the two states of polarization are obtainable from one another by rotation through 45°. Referring to Eq. (9.7.11) it is easy to see the effect which a polarized plane gravitational wave has on the strain tensor of a medium with which it interacts. This effect is schematically indicated in Fig. (9-2) for a plane wave propagating into the page. The wave induces an oscillating transverse quadrupole moment in an otherwise spherically symmetric object. From the figure it is clear that a rotation through 90° yields again the same state of polarization.

Using the Fourier decomposition of the function $D(x - x')$, namely,

$$\begin{aligned} D(x - x') &= (2\pi)^{-4}\left(\int_{C^+} - \int_{C^-}\right)(k_\mu k^\mu)^{-1} e^{ik_\nu(x^\nu - x'^\nu)}\, d^4k \\ &= -i\left[(2\pi)^{-3}\int \tfrac{1}{2}\, e^{ik_\mu(x^\mu - x'^\mu)}(d^3\underset{\sim}{k}/k^0) - \text{h. c.}\right] , \end{aligned} \tag{9.7.50}$$

together with the identity

$$(\underset{\sim}{e}_1\underset{\sim}{e}_1 - \underset{\sim}{e}_2\underset{\sim}{e}_2)_{ab}(\underset{\sim}{e}_1\underset{\sim}{e}_1 - \underset{\sim}{e}_2\underset{\sim}{e}_2)_{cd} + (\underset{\sim}{e}_1\underset{\sim}{e}_2 + \underset{\sim}{e}_2\underset{\sim}{e}_1)_{ab}(\underset{\sim}{e}_1\underset{\sim}{e}_2 + \underset{\sim}{e}_2\underset{\sim}{e}_1)_{cd}$$

$$\equiv \delta^T{}_{ac}\delta^T{}_{bd} + \delta^T{}_{ad}\delta^T{}_{bc} - \delta^T{}_{ab}\delta^T{}_{cd} , \tag{9.7.51}$$

where

$$\delta^T{}_{ab} \equiv (\underset{\sim}{e}_1\underset{\sim}{e}_1 + \underset{\sim}{e}_2\underset{\sim}{e}_2)_{ab} \equiv \delta_{ab} - k_a k^{-2} k_b , \tag{9.7.52}$$

it is easily verified that the commutation relations (9.7.41) and (9.7.42) are satisfied if and only if

$$[a_I(\underset{\sim}{k}) , a_I^\dagger(\underset{\sim}{k}')] = [a_{II}(\underset{\sim}{k}) , a_{II}^\dagger(\underset{\sim}{k}')] = \delta(\underset{\sim}{k} - \underset{\sim}{k}') , \tag{9.7.53}$$

while the commutators of all other pairs of amplitudes vanish. The amplitudes a_I, a_{II}, $a_I^\dagger$, $a_{II}^\dagger$ are recognized as graviton annihilation and creation operators respectively.

Creation and annihilation operators for states of definite spin are obtained from the amplitudes a_I, a_{II}, etc. by the unitary transformation:

$$a_\pm \equiv 2^{-\frac{1}{2}} (a_I \mp i\, a_{II}) . \tag{9.7.54}$$

Under a rotation in the positive direction through an infinitesimal angle $\delta\phi$ about the vector $\underset{\sim}{e}_3$, the vectors $\underset{\sim}{e}_i$ suffer the changes

$$\delta\underset{\sim}{e}_i = \underset{\sim}{e}_3 \times \underset{\sim}{e}_i \, \delta\phi , \tag{9.7.55}$$

and hence

$$\left.\begin{aligned} \delta(\underset{\sim}{e}_1\underset{\sim}{e}_1 - \underset{\sim}{e}_2\underset{\sim}{e}_2) &= 2\,(\underset{\sim}{e}_1\underset{\sim}{e}_2 + \underset{\sim}{e}_2\underset{\sim}{e}_1)\delta\phi , \\ \delta(\underset{\sim}{e}_1\underset{\sim}{e}_2 + \underset{\sim}{e}_2\underset{\sim}{e}_1) &= -\,2\,(\underset{\sim}{e}_1\underset{\sim}{e}_1 - \underset{\sim}{e}_2\underset{\sim}{e}_2)\delta\phi . \end{aligned}\right\} \tag{9.7.56}$$

Therefore, since

$$[s, a_I] = 2\,i\,a_{II} \;, \qquad [s, a_{II}] = -2\,i\,a_I \;, \tag{9.7.57}$$

where

$$s \equiv 2\,(a_+^\dagger a_+ - a_-^\dagger a_-) \;, \tag{9.7.58}$$

it follows that the spin operator must be given by

$$\underset{\sim}{\sigma} \equiv s\,\underset{\sim}{e}_3 \;, \tag{9.7.59}$$

for it acts as the generator of infinitesimal rotations according to the fundamental law for angular momentum:

$$\begin{aligned} [\sigma_3 \,,\, \underset{\sim}{E}^{\text{free}}(k)] &= i\,\delta\,\underset{\sim}{E}^{\text{free}}(k) \;, \\ [\sigma_3 \,,\, \underset{\sim}{H}^{\text{free}}(k)] &= i\,\delta\,\underset{\sim}{H}^{\text{free}}(k) \;, \end{aligned} \tag{9.7.60}$$

$\underset{\sim}{E}^{\text{free}}(k)$ and $\underset{\sim}{H}^{\text{free}}(k)$ being the total Fourier amplitudes of the gravitational field. Since gravitational waves are purely transverse it is possible to generate rotations only about the vector $\underset{\sim}{e}_3$, and hence the spin is restricted to be parallel or antiparallel to $\underset{\sim}{e}_3$. The factor 2 in Eq. (9.7.58) identifies the gravitational field as a spin-2 field.

The amplitudes a_I, a_{II}, etc. may be used to define a "total" energy and momentum for the linearized free gravitational field:

$$H \equiv \int k^0 (a_I^\dagger a_I + a_{II}^\dagger a_{II})\, d^3\underset{\sim}{k} \;, \tag{9.7.61}$$

$$\underset{\sim}{P} \equiv \int \underset{\sim}{k} (a_I^\dagger a_I + a_{II}^\dagger a_{II})\, d^3\underset{\sim}{k} \;. \tag{9.7.62}$$

It is to be emphasized, however, that the validity of these definitions is strictly limited to states in which the field is everwhere weak inside a macroscopically flat region of interest and effectively vanishes for a considerable distance outside. Furthermore, although the definitions are then applicable to the total energy and momentum inside the region in question [by imposing periodic boundary conditions on the region and replacing the integrals in (9.7.61) and (9.7.62) by sums] it is on the other hand, impossible to introduce the concept of a local distribution of energy and momentum which is group-invariant. For, although the amplitudes a_{I}, a_{II}, etc. are Fourier transforms of the group-invariants $\underset{\sim}{E}^{free}$ and $\underset{\sim}{H}^{free}$, and hence permit H and $\underset{\sim}{P}$ to be re-expressed as integrals over all space, nevertheless, because of the necessity of inverting the factor k^2 which appears in the integrands of expressions (9.7.45) and (9.7.46), the integrands of the spatial integrals---which would normally be identified as the energy and momentum densities respectively---cannot be expressed in terms of the local geometry of space-time but become non-local functionals of $\underset{\sim}{E}^{free}$ and $\underset{\sim}{H}^{free}$.

The only strictly local quantities presently known which satisfy a field-equation-dependent differential conservation law analogous to the laws of conservation of energy and momentum in Lorentz invariant theories are the components of a fourth rank tensor discovered by Bel (1959) and Robinson (1959). Since the conservation law which it satisfies is completely covariant and independent of the weak-field approximation this tensor is of undoubted importance, although its physical significance is not yet well understood in concrete terms.

(9.3) Measurability of the gravitational field.

It has been emphasized by Bohr and Rosenfeld (1933) that the classical description of a field in terms of components (in our case, E_{ab} and H_{ab}) at each space-time point becomes, in the quantum theory, an idealization having only a formal applicability. Since the commutators of the quantized field components involve singular functions (e.g., $D(x - x')$ and its derivatives), unambiguous statements can be deduced from the formalism only for averages of the field components over finite space-time regions. Our first problem will be to find suitable devices for measuring such field averages and to examine the kinds of averages to which these devices lead.

We begin with the description of a convenient way to measure (conceptually) an average of the component E_{11}, noting that the same measurement repeated in a sufficient number of differently oriented quasi-Cartesian coordinate systems---that is to say, with the measurement device itself placed in different orientations---suffices for the determination of similar averages of all the components E_{ab}. Thus, if the measurement is performed in a coordinate system of which the x'-axis has direction cosines α_1, α_2, α_3 with respect to the original system, then an average is obtained for the quantity $\alpha_a\alpha_b E_{ab}$ in the original system. It is easy to see that averages of all the E_{ab} in the original system may thereby be inferred by making six appropriate different choices of the α_a.[27]

For the measurement of a space-time average of E_{11} we must insert some kind of a "test body" into the field. The simplest body which we can use for this purpose is the elastic medium itself, which, together with its clocks, was introduced originally for the purpose of defining a local coordinate system. The spatial boundaries of the medium may be taken to coincide with those of the averaging domain, since the coordinate system is needed only there and nowhere else. The quasi-rigidity of the medium, together with the weak-field situation, which must be assumed, insures that the coordinate system remains permanently quasi-Minkowskian.[28] For simplicity the spatial volume occupied by the medium will be taken as a quasi-cube having sides of unstrained length L oriented along the coordinate axes. At a certain instant---marking the temporal beginning of the space-time averaging domain---the internal constitution of the medium is abruptly and simultaneously (as determined by the clocks) altered throughout, in such a way that a simple application of Eq. (9.7.11) can then be made to obtain a determination of the average value of E_{11} over the medium and over the length of time during which the medium retains its altered constitution. The nature of the most suitable alteration will now be examined.

We first consider an alteration which is physically inadmissible, but which will nevertheless lead us to a correct analysis. We imagine that the elastic moduli c_{abcd} suddenly become modified in such a way that the medium no longer supports short wavelength oscillations but becomes what we may call <u>strain-rigid</u>---that is, the only internal motion which it can execute is a <u>uniform</u> strain in the x^1-direction, which is itself unhindered by elastic restoring forces. This situation is somewhat picturesquely illustrated by the device shown in Fig. (9-3). Its physical inadmissability resides, of course, in the fact that because of the finite propagation speed of all forces a body can no more be strain-rigid than it can be truly rigid. Leaving aside this defect for the moment, however, let us see how such a hypothethical device might be used. Let us assume that the intrinsic coordinate system defined by the test body has been adjusted so

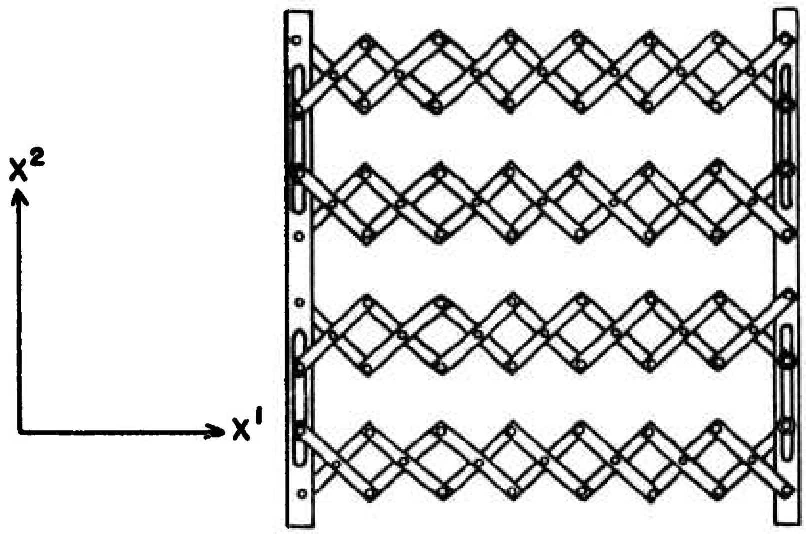

Fig. 9-3. A strain-rigid test body.

that at the initial instant g_{11} is equal to unity, to an accuracy limited only by the zero point oscillations. Lengths in the x^1-direction are thus read off directly from the coordinate x^1 at this instant. At subsequent instants the component particles of the test body will have suffered displacements δz^a given by [29]

$$\delta z^a = \delta^a{}_1 s_{11} x^1 \quad , \tag{9.8.1}$$

where s_{11} is the diagonal component of the (now uniform) strain tensor in the x^1-direction and where the origin of coordinates is chosen at the center of the body. Assuming uniform mass density ρ_0 we have, from Eq.(9.7.11),

$$\ddot{s}_{11} = E_{11} - \rho_0^{-1} t_{1a,a1} \quad , \tag{9.8.2}$$

which expresses the temporal behavior of the strain resulting from the action of the space-time curvature and the internal stresses. We do not, of course, know in detail what the internal stresses are. But we know that they must adjust themselves in such a way that the body as a whole undergoes strain-rigid motion. The forces involved are evidently forces of constraint which can do no net work in a virtual displacement. This condition may be expressed in the form

$$0 = \int_V t_{ab,b}\,\delta(\delta z^a)\,d^3\underset{\sim}{x} = (\delta s_{11}) \int_V t_{1b,b}\, x^1 d^3\underset{\sim}{x} , \tag{9.8.3}$$

where $V(=L^3)$ denotes the volume occupied by the test body. From this it follows, through an integration by parts, that

$$\int_V (x^1 + \tfrac{1}{2} L)(\tfrac{1}{2} L - x^1)\, t_{1a,a1}\, d^3\underset{\sim}{x} = 2\int_V x^1 t_{1a,a}\, d^3\underset{\sim}{x} = 0 \, , \tag{9.8.4}$$

and hence

$$\ddot{s}_{11} = \bar{E}_{11} \quad , \tag{9.8.5}$$

$$\bar{E}_{11} \equiv 6\, L^{-5} \int_V (x^1 + \tfrac{1}{2} L)(\tfrac{1}{2} L - x^1)\, E_{11}\, d^3\underset{\sim}{x} . \tag{9.8.6}$$

The quantity which the strain-rigid test body measures is therefore seen to be a weighted average of the field component E_{11} over the volume occupied by the test body. The weighting factor $(x^1 + \frac{1}{2} L)(\frac{1}{2} L - x^1)$ is parabolic, going to zero at the extremities of the body in the x^1-direction.

To test the formalism of the quantum theory of geometry by means of measurements we nust assume that the apparatus (in this case the test body) obeys the Uncertainty Principle. Poisson brackets for a strain-rigid test body can be obtained by the general Green's-function techniques outlined in the early sections of this chapter. Conventional methods, however, suffice for this simple example. Since the internal forces are pure constraints the Lagrangian is just the total kinetic energy:

$$L = \frac{1}{2}\int_V \rho_0 (\delta \dot{z}^a)^2\, d^3\underset{\sim}{x} = \frac{1}{24} ML^2 (\dot{s}_{11})^2 \quad . \tag{9.8.7}$$

M is the total mass of the test body. The variable conjugate to the strain, which we may call the strain momentum, is given by

$$\pi = \partial L/\partial \dot{s}_{11} = \frac{1}{12} ML^2 \dot{s}_{11} \quad . \tag{9.8.8}$$

The accuracy of a simultaneous fixing of s_{11} and π is limited by the uncertainty relation

$$\Delta s_{11} \Delta \pi \sim 1 \quad . \tag{9.8.9}$$

A measurement of π at the beginning and end of a time interval T yields a space-time average of E_{11}. Denoting the measured values by π' and π''

respectively, we have

$$\pi'' - \pi' = \frac{1}{12} ML^2 \int_T \ddot{s}_{11}\, dx^0 = \frac{1}{12} ML^2 T \bar{\bar{E}}_{11} , \tag{9.8.10}$$

$$\bar{\bar{E}}_{11} \equiv T^{-1} \int_T \bar{E}_{11}\, dx^0 \equiv 6\, L^{-5} T^{-1} \int_{VT} (x^1 + \tfrac{1}{2} L)(\tfrac{1}{2} L - x^1) E_{11} d^4x . \tag{9.8.11}$$

Equation (9.8.10) may be solved to express $\bar{\bar{E}}_{11}$ in terms of the "experimental data" π' and π''. The limitation which the uncertainty relation (9.8.9) imposes on the accuracy of this measurement is evidently given by

$$\Delta \bar{\bar{E}}_{11} \sim \frac{12}{ML^2 T\, \Delta s_{11}} , \tag{9.8.12}$$

which, for every value of Δs_{11} no matter how small, can be made arbitrarily small by the choice of a sufficiently large value of M. On the other hand, M is limited by the weak-field condition

$$M \ll L , \tag{9.8.13}$$

and here we encounter a situation which has no analog in the measurement problem of electrodynamics which Bohr and Rosenfeld considered. It turns out, as a result of the complete analysis of the sources of uncertainty in the measurement of $\bar{\bar{E}}_{11}$ which are present in addition to that expressed by (9.8.12), that this situation has a more fundamental significance than the mere breakdown of an approximation method. We now examine these additional sources of uncertainty.

To begin with it is necessary to point out that the boundary of the space-time averaging domain is not defined with infinite precision by the experimental arrangement. In addition to the zero point oscillations there are two sources of uncertainty in this boundary: (1) an uncertainty in the range of the time interval T due to the fact that the measurements of the strain momenta π' and π'' actually occupy finite time intervals Δt; (2) an uncertainty in the spatial boundaries due to the strain suffered by the test body during the measurement process. As for the uncertainty in time it is clear that we must assume

$$\Delta t \ll T \quad , \tag{9.8.14}$$

if Eq. (9.8.10) is to be at all usable. On the other hand, it must be borne in mind that if Δt is taken too small, the uncertainty in the total mass of the clocks will become great enough to violate the weak-field condition on which Eq. (9.8.10) is based. It will turn out that these conflicting requirements can be balanced only if a fundamental limitation is imposed on the size of allowable measurement domains.

The uncertainty in the spatial boundaries will remain within tolerable limits only if s_{11} remains small compared to unity throughout the interval T. From Eq. (9.8.5) this is seen to impose the requirement

$$E_{11}T^2 \ll 1 \ . \tag{9.8.15}$$

Following Bohr and Rosenfeld it will be convenient to confine our attention to the case

$$T < L \quad , \tag{9.8.16}$$

which permits an approximate approach to a limiting situation analogous to non-relativistic particle mechanics, in which a strict temporal order can be assigned to measurement sequences. Equation (9.8.15) then becomes simply a special case of the general weak field condition

$$RL^2 \ll 1 , \tag{9.8.17}$$

where R denotes the magnitude of a typical component of the Riemann tensor. In the electromagnetic measurement problem Bohr and Rosenfeld kept the spatial boundary within tolerable limits by choosing the mass of the test body sufficiently large. In the present problem the same result is achieved by the weak field condition, whereas the mass now plays the role previously played by the charge. This is, of course, quite reasonable in view of the physical significance of mass in the theory of gravitation. It will become evident, furthermore, that the lack of a freely adjustable charge-to-mass ratio, which has sometimes been predicted as an obstable to the measurability analysis for gravitational fields (see, e.g., Rosenfeld, 1957), in fact poses no obstacle other than a limitation on the smallness of allowable measurement domains.

The magnitude of the errors introduced by the imprecision of the boundaries of the space-time averaging domain is proportional to the absolute magnitude of E_{11} itself, and can surpass all limits as E_{11} becomes arbitrarily large. As Bohr and Rosenfeld have empmphasized, however, this circumstance corresponds only to the general limitation on all physical measurements, whereby a knowledge of the order of magnitude of the effect to be expected is always necessary for the choice of appropriate measuring instruments. In the measurability problem we are interested in fields which are so weak as to place us clearly in the quantum domain. Such fields are those for which fluctuation phenomena become significant. The strength of the fluctuations involved in a given measurement is effectively determined by the magnitude of the commutator taken between a typical pair of field quantities similar to the field quantity being measure. A typical situation, and the one which must be studied in carrying out the complete analysis of the measurability problem, involves the measurement of two field components averaged respectively over two overlapping space-time domains. Here the test bodies themselves must be assumed to interpenetrate one another without interaction, although prior to their respective time intervals T they should be bound firmly together so that they have no relative velocity at the beginning of the measurement. A local quasi-Cartesian coordinate system may be introduced which embraces both bodies at once, or else, according to convenience, two such systems may be introduced, each centered on one of the bodies and oriented parallel to its edges, the two bodies being assumed to be quasi-cubes of comparable volume. The bodies will be distinguished by the labels I and II . If the direction cosines of the x^1-axes of the two bodies with respect to the common coordinate system are $\alpha_a{}^{I}$ and $\alpha_a{}^{II}$ respectively, then the commutator of the field averages measured by the two bodies over their respective time intervals is given by

$$[\bar{\bar{E}}^{I}, \bar{\bar{E}}^{II}] = \frac{1}{4} i \int d^4x \int d^4x' \, \alpha_a{}^{I}\alpha_b{}^{I}\alpha_c{}^{II}\alpha_d{}^{II} W_E{}^{I}(x)\, W_E{}^{II}(x') \times (\delta^T{}_{ac}\delta^T{}_{bd} + \delta^T{}_{ad}\delta^T{}_{bc} - \delta^T{}_{ab}\delta^T{}_{cd})\nabla^4 D(x - x'). \qquad (9.8.18)$$

Here the subscripts 11 have been dropped on the $\bar{\bar{E}}$, and weight functions $W_E{}^{I}$ and $W_E{}^{II}$ appropriate to the measurement of the field components $\bar{\bar{E}}_{11}$ for the two bodies respectively have been introduced. In a coordinate system centered on one of the test bodies and oriented with it, the weight function is given by

$$W_E(x) \equiv 6\, L^{-5}T^{-1}(x^1 + \tfrac{1}{2} L)(\tfrac{1}{2} L - x^1)\theta(x^1 + \tfrac{1}{2} L)\theta(\tfrac{1}{2} L - x^1)\theta(x^2 + \tfrac{1}{2}L)\theta(\tfrac{1}{2}L - x^2) \times \theta(x^3 + \tfrac{1}{2} L)\theta(\tfrac{1}{2} L - x^3)\theta(x^0)\theta(T - x^0) , \qquad (9.8.19)$$

θ denoting, as usual, the step function, and the origin of time being taken at the beginning of the measurement interval.

By introducing the explicit form (9.7.43) of the transverse-field projection operator, performing a number of integrations by parts, making use of the wave equation (9.7.44) satisfied by $D(x - x')$, and finally splitting $D(x - x')$ into its advanced and retarded parts while taking note of the reciprocity relations satisfied by these parts, it is not hard to show that Eq.(9.8.18) may be rewritten in the form

$$[\bar{\bar{E}}^{I}, \bar{\bar{E}}^{II}] = i\,(A^{I,II} - A^{II,I}) \quad , \tag{9.8.20}$$

where

$$\begin{aligned} A^{I,II} = \frac{1}{4}\int d^4x \int d^4x'\, W_E{}^{II}(x')\, D^-(x'-x)\, \alpha_a{}^I \alpha_b{}^I \alpha_c{}^{II} \alpha_d{}^{II} [(2\delta_{ac}\delta_{bd} \\ - \delta_{ab}\delta_{cd}) W_E{}^I{}_{,0000}(x) - 4\,\delta_{ac} W_E{}^I{}_{,bd00}(x) + \delta_{ab} W_E{}^I{}_{,cd00}(x) \\ + \delta_{cd} W_E{}^I{}_{,ab00}(x) + W_E{}^I{}_{,abcd}(x)\,] \,. \end{aligned} \tag{9.8.21}$$

The verification of the uncertainty relation

$$\Delta\bar{\bar{E}}^{I}\, \Delta\bar{\bar{E}}^{II} \sim |A^{I,II} - A^{II,I}| \tag{9.8.22}$$

which follows from Eq. (9.8.20) will be the main task of this section.

The magnitued of the quantity $|A^{I,II} - A^{II,I}|$ in the case of partial overlap of the space-time regions $V_I T_I$ and $V_{II} T_{II}$ may be estimated by inserting the Fourier decomposition (9.7.50) of the propagation function into (9.8.18) and performing the space-time integrals first and the momentum integration last. Assuming $L_I \sim L_{II} \sim L$ and $T_I \sim T_{II} \sim T$ one obtains, for the square root of this estimate, the critical field strength

$$R_{crit} = \begin{cases} L^{-3} & \text{when} \quad L \sim T \\ L^{-\frac{5}{2}} T^{-\frac{1}{2}} & \text{when} \quad L \gg T \end{cases} \tag{9.8.23}$$

below which quantum phenomena become important. The value $R_{crit} = L^{-3}$ being always the smaller of the two given in (9.8.23), we shall be safe in using it in all cases. The criterion of accuracy for testing the formalism in the quantum domain is then

$$\Delta \bar{\bar{E}} \sim \lambda R_{crit} = \lambda L^{-3} \tag{9.8.24}$$

where

$$\lambda \ll 1 \tag{9.8.25}$$

for all sources of uncertainty $\Delta\bar{\bar{E}}_{11}$.

The critical field strength R_{crit} represents the magnitude of the quantum fluctuations. These fluctuations must themselves satisfy the weak-field condition (9.8.17) if the experimental arrangement is to have any utility. From this it follows that we must have

$$L \gg 1 . \tag{9.8.26}$$

A still more stringent limitation is, in fact, required if complete consistency between formalism and measurement is to be achieved. For, returning to Eq. (9.8.12) and taking note of the necessary restriction

$$\Delta s_{11} \ll 1 , \tag{9.8.27}$$

we see that the mass of the test body must satisfy

$$M \sim \frac{12}{L^2 T \, \Delta\bar{\bar{E}}_{11} \Delta s_{11}} \sim 12 \frac{1}{\lambda} \frac{1}{\Delta s_{11}} \frac{L}{T} \ggg 1 . \tag{9.8.28}$$

if an accuracy is to be attained sufficient to test the formalism in the quantum domain. From condition (9.8.13) it therefore follows that

$$L \ggg\gg 1 . \tag{9.8.29}$$

We note that if experimentally known elementary particles (masses between 10^{-22} and 10^{-18}) are used in the construction of the test body, condition (9.8.28) implies that a very large number will be needed. This, of course, fits well with the representation of the test body as a continuous medium. Furthermore, it underscores, as in the case of electrodynamics, the inappropriateness of attempts to use individual point particles as test bodies in making accurate field measurements (Landau and Peierls, 1931; Anderson, 1954; Salecker, 1957). Indeed, it is only by using extended test bodies that uncertainties due to the forces of radiation reaction may be ignored. In the present example these uncertainties arise from uncertainties in the stress-energy density of amount

$$\Delta T \sim ML^{-3}\Delta s_{11} \tag{9.8.30}$$

[see also Eqs. (9.8.59), (9.8.60) and (9.8.63)] which themselves arise from the process of measuring the initial and final values, π' and π'' , of the strain momentum during the time intervals Δt . During such time intervals the uncertainty (9.8.30) gives rise to an uncertainty in the Riemann tensor of comparable magnitude, which in turn produces an additional uncertainty in π' and π'' beyond that determined by the uncertainty relation (9.8.9), namely

$$\delta\pi \sim \frac{1}{12} ML^2 \, \Delta t \, \Delta T \sim \frac{1}{12} M^2L^{-1} \, \Delta t \, \Delta s_{11} . \tag{9.8.31}$$

This additional uncertainty may be neglected in comparison with $\Delta\pi$ by choosing Δt so small that

$$1 \gg \delta\pi/\Delta\pi \sim \frac{1}{12} M^2 L^{-1} \Delta t (\Delta s_{11})^2$$

$$\sim 12\, L^{-5} T^{-2} \Delta t (\Delta \bar{\bar{E}}_{11})^{-2} \sim 12 \frac{1}{\lambda^2} \frac{L}{T} \frac{\Delta t}{T} . \qquad (9.8.32)$$

We have already pointed out, however, that Δt must not be chosen so small that the mass M violates the weak-field condition (9.8.13). It is easy to see that this caution again leads to the limitation (9.8.29) on the smallness of allowable measurement domains.

Before proceeding to the verification of the uncertainty relation (9.8.22) it is still necessary to investigate the physical means by which the strain momentum is measured and the sources of error to which the procedures involved unavoidably give rise. In order to do this it will first of all be necessary to drop the untenable original assumption of strain-rigidity for the test body. It is apparent from the foregoing discussion that it is only the degree of uncontrollability Δs_{11} in the strain which need be uniform. This, however, can be arranged by a prescription for the measurement process which does no violence to the relativity principle. At the beginning of the interval T, instead of adjusting the elastic moduli so as to insure strain-rigidity we simply let them all fall abruptly to zero. One may imagine this change to be brought about by a loosening of the coupling between the consituent particles, which transforms the medium into an ensemble of free particles. Because of the retardation of forces the actual decoupling process must occupy an interval of time at least as big as the interparticle spacing ℓ. If the original elastic moduli are chosen big enough to make the sound velocities approach the velocity of light this time interval will be of the order of the period of the short wavelength $(\sim \ell)$ oscillations of the constituent particles. The time intervals Δt for the subsequent measurements of the strain momentum may also be taken of comparable magnitude.

The zero point coupling energy, which was shown in Section 5 to be small compared to the total mass, does not, of course, simply disappear, but must be accounted for. It may be imagined as temporarily stored within the constituent particles themselves. However, it is not necessary to be precise about the mechanism for accomplishing this. The lack of a detail prescription in this regard is not to be understood as implying any fundamental uncertainty in the Riemann tensor arising from this source. It is enough to know that such a prescription is in principle possible.

The measurement of the strain momentum is most easily carried out by a simple generalization of the Doppler shift technique employed by Bohr and Rosenfeld. At a given moment prior to the interval T a light source which is located within a relatively small region at the center of the test body, and which may itself be regarded as a component of the test body, emits a bundle of electromagnetic radiation of duration no greater than Δt. By means of suitably placed mirrors portions of this radiation are progressively delayed---for example, by temporary trapping in a central slab at right angles to the x^1 axis---so that the radiation bundle becomes an "extended projectile" which strikes all portions of the test body at the same instant immediately after the coupling between the constituent particles is removed. The state of the radiation bundle just prior to impact with the test body may be described in terms of photon density, which is arranged so as to be proportional to the magnitude of the coordinate x^1 and independent of

x^2 and x^3, with the photons themselves propagating parallel to the x^1 axis, in the positive direction for positive values of x^1 and in the negative direction for negative values of x^1. By means of mirrors attached to the constituent particles of the test body the radiation bundle, as a result of the impact, is reflected back to the central source where it is analyzed spectroscopically for a determination of the Doppler shift. Although the radiation bundle is here described in terms of photons it is to be emphasized that these photons are emitted coherently, and there is never any question of measuring the momenta (relative to the central plane) of the constituent particles individually. Only the total strain momentum is measured, for only then can the minimum uncertainties expressed by (9.8.9) be achieved. Under these circumstances the uncertainty ΔE in the total energy of the radiation bundle will satisfy the relation

$$\Delta E \, \Delta t \sim 1 \ . \tag{9.8.33}$$

The coherence of the emission process furthermore implies that the quantity ΔE also represents the uncertainty in the <u>total</u> mass of the clocks which are needed to time the emission process. In fact, the emission process may be regarded as a transfer of energy from the clocks to the radiation field. Similarly the subsequent removal of the interparticle coupling may be regarded as a coherent transfer of energy from the coupling mechanisms to the clocks, having a comparable uncertainty in total magnitude.

Since the strain is no longer required to be uniform during the interval T the definition (9.8.8) of the strain momentum must be modified. The appropriate generalization is simply

$$\pi \equiv \sum_{\tau} x^1{}_{\tau}\, p_{1\tau} \tag{9.8.34}$$

where the summation is extended over the constituent particles of the test body, $x^1{}_\tau$ and $p_{1\tau}$ being the x^1-components of position and momentum of the τ th particle respectively. It is easy to see that this definition reduces to (9.8.8) when the strain is uniform, with $p_{1\tau} = m\, x^1{}_\tau \dot{s}_{11}$. To see, furthermore, that Eq. (9.8.10) retains its validity it is only necessary to note that we now have $s_{11} = \delta z_{1,1}$ and that Eq.(9.8.2) is replaced by

$$\delta\ddot{z}_{1,1} = E_{11} \ . \tag{9.8.35}$$

Hence

$$\bar{E}_{11} = 6\, L^{-5}\int_V (x^1 + \tfrac{1}{2} L)(\tfrac{1}{2} L - x^1)\ \delta\ddot{z}_{1,1}\ d^3\underset{\sim}{x}$$

$$= 12\, L^{-5}\int_V x^1\ \delta\ddot{z}_1\ d^3\underset{\sim}{x}\ , \tag{9.8.36}$$

$$\bar{\bar{E}}_{11} = T^{-1}\int_T \bar{E}_{11}\ dx^0 = 12\, M^{-1}L^{-2}T^{-1}\int_V \rho_0\ x^1(\delta\dot{z}_1{}'' - \delta\dot{z}_1{}')\ d^3\underset{\sim}{x}$$

$$= 12\, M^{-1}L^{-2}T^{-1} \sum_\tau x^1{}_\tau(p_{1\tau}{}'' - p_{1\tau}{}')$$

$$= 12\, M^{-1}L^{-2}T^{-1}(\pi'' - \pi'). \tag{9.8.37}$$

In order to verify that the prescribed radiation bundle actually measures the strain momentum (9.8.34) it is necessary to compute the exchange of energy and momentum between photons and particles. We assume that all the photons have as nearly as possible the same angular frequency ω_0. Their actual frequencies ω' will generally differ from ω_0 by amounts of order ΔE

satisfying condition (9.8.33). Because of their coherence the uncertainty in their total energy will, as has already been mentioned, also be equal to ΔE. In the discussion which immediately follows we use the prime and double prime to denote respectively "before" and "after" the collision between the particles and photons. For positive values of $x^1{}_\tau$ we have

$$p_{1\tau}'' - p_{1\tau}' = \sum_{n_\tau} (\omega' + \omega''), \tag{9.8.38}$$

$$\frac{1}{2} m^{-1}(p_{1\tau}''^2 - p_{1\tau}'^2) = \sum_{n_\tau} (\omega' - \omega''), \tag{9.8.39}$$

where n_τ is the number of photons striking the τ th particle and where the velocities imparted to the particles are assumed to be nonrelativistic. The latter assumption will be valid provided

$$n_\tau \omega_0 \ll m \quad , \tag{9.8.40}$$

which requires that the total photon energy shall remain small compared to the rest energy of the test body. From Eqs. (9.8.38) and (9.8.39) one obtains

$$p_{1\tau}' = m \frac{\sum_{n_\tau} (\omega' - \omega'')}{\sum_{n_\tau} (\omega' + \omega'')} - \frac{1}{2}\sum_{n_\tau} (\omega' + \omega'') \ , \tag{9.8.41}$$

and for negative values of $x^1{}_\tau$ the same expression is obtained with the opposite sign. The condition (9.8.40) insures that the mean frequency ω_0 of the radiation bundle will be large compared to the Doppler shifts $\omega' - \omega''$.

If it is also taken large compared to the frequency spread ΔE we may write, to good approximation,

$$p_{1\tau}' = \frac{1}{2} m (n_\tau \omega_0)^{-1} \sum_{n_\tau} (\omega' - \omega'') - n_\tau \omega_0 , \; (x^1{}_\tau > 0) \qquad (9.8.42)$$

and hence

$$\pi' = \sum_\tau \frac{1}{2} m (n_\tau \omega_0)^{-1} |x^1{}_\tau| \sum_{n_\tau} (\omega' - \omega'') - \sum_\tau |x^1{}_\tau| n_\tau \omega_0 . \qquad (9.8.43)$$

Since n_τ is assumed to be proportional to $|x^1{}_\tau|$ the first term of Eq. (9.8.43) is simply proportional to $\sum_\tau \sum_{n_\tau} (\omega' - \omega'')$, and hence the mean total Doppler shift, which may be determined immediately from the spectral analysis of the reflected radiation bundle, gives a direct measure of the strain momentum.

The factor of proportionality between n_τ and $|x^1{}_\tau|$ is readily obtained from the observation that as a result of the collisions with the photons the constituent particles of the test body will, during the collision time Δt, undergo uncontrollable displacements in addition to the displacements which they would undergo in the absence of the measurement, of order

$$\Delta z_{1\tau} = m^{-1} (p_\tau'' - p_\tau') \Delta t . \qquad (9.8.44)$$

Imposing the uniform strain requirement $\Delta z_{1\tau} = x^1{}_\tau \Delta s_{11}$ and taking note of Eq. (9.8.38), we therefore infer

$$n_\tau = \frac{1}{2} m \omega_0^{-1} (\Delta s_{11}/\Delta t) |x^1{}_\tau| \ll m L \Delta s_{11} . \qquad (9.8.45)$$

From Eq. (9.8.43) it then follows that the uncertainty in the strain momentum measurement is given by

$$\Delta\pi = (\Delta t/\Delta s_{11})\, \Delta \sum_{T} \sum_{n_T} (\omega' - \omega''). \tag{9.8.46}$$

Since the energy of the final radiation bundle is measured with arbitrary precision in the spectral analysis, the uncertainty in the mean total Doppler shift is due entirely to the initial energy uncertainty:

$$\Delta \sum_{T} \sum_{n_T} (\omega' - \omega'') = \Delta E. \tag{9.8.47}$$

Equation (9.8.46) together with the uncertainty relation (9.8.33) therefore leads again to (9.8.9), showing that the conjugate relationship between s_{11} and π is maintained even though the test body is no longer strain-rigid.

We may note that the condition $\omega_0 >> \Delta E$ together with (9.8.40) further reinforces the limitation (9.8.29) on the smallness of allowable measurement domains. We have

$$L >> M >> \sum_{T} n_T \omega_0 >> \Delta t^{-1} \sum_{T} n_T \quad . \tag{9.8.48}$$

On the other hand, we must evidently have

$$\sum_{T} n_T >>> 1 \quad , \tag{9.8.49}$$

which together with (9.8.32) leads to

$$L >>> \frac{1}{\lambda}\ \frac{L}{T}\ \left(12 \sum_{T} n_T\right)^{\frac{1}{2}} >>> 1 \quad . \tag{9.8.50}$$

Under restrictions of such stringency the composition of the radiation bundle can easily be arranged so that the condition (9.8.49) is compatible with (9.8.45), which, in combination with (9.8.12) and (9.8.24) yields

$$\sum_T n_T \ll ML\,\Delta s_{11} \sim 12\,\frac{1}{\lambda}\,\frac{L}{T}\,L\,. \tag{9.8.51}$$

Before completing the detailed description of the measurement process let us enumerate the sources of uncertainty in the field measurement which remain to be discussed. These are: (1) the disturbance in the field produced by the uncontrollable component of the stress-energy density resulting from the strain uncertainty Δs_{11} associated with the measurement of π'; (2) the disturbance produced by the radiation bundle; (3) the boundary uncertainties arising from the zero point oscillations of the test body; (4) the disturbances produced by compensation mechanisms which will presently be introduced. Of these we shall show that only the first is significant under the limitations which have already been imposed on the structure and dimensions of the test body and the parameters of the measurement.

As far as the radiation bundle is concerned the only uncertainty which its emission produces in the gravitational field is that due to the uncertainty ΔE in the energy transferred to it from the group of clocks constituting the radiation source at the center of the test body. The main effect which the emission of the radiation bundle has on the gravitational field can be computed in advance---and hence allowed for---from a detailed knowledge (available in principle) of the structure of the radiation source and the arrangement of the various mirrors. This point is important since the same argument also applies to the test body itself. As has been pointed out by Heitler (1954) it is only the uncontrollable motion of the test body which gives rise to an uncertainty in the field. In the present case this motion produces a quadrupole change in the stress-energy tensor, of magnitude given by (9.8.30). The monopole field of the test body, on the other hand, is already known, and does not need to be compensated for---by introducing, for example, charges of opposite sign as Bohr and Rosenfeld did for the measurement of the electromagnetic field. This is fortunate in the present case, since negative masses do not exist.[30]

The uncertainty in the energy exchange between radiation source and radiation bundle gives rise to an uncertainty in the stress-energy density of order

$$\delta T = L^{-3} \Delta E \,, \tag{9.8.52}$$

which is also the same as the order of the uncertainty involved in the energy exchange between the interparticle coupling mechanisms and the clocks, which takes place when the elastic moduli are altered. The ratio of the uncertainties (9.8.52) and (9.8.30) may be expressed in the forms

$$\frac{\delta T}{\Delta T} \sim \frac{\Delta E}{M \Delta s_{11}} \sim \frac{1}{12} L^2 T \frac{\Delta \bar{\bar{E}}_{11}}{\Delta t} \sim \frac{1}{12} \lambda L^{-1} \frac{T}{\Delta t} \gg \frac{1}{\lambda} \frac{1}{T} \,, \tag{9.8.53}$$

the final inequality following from (9.8.32). It is seen that the uncertainty δT may be neglected im comparison with ΔT provided

$$T \ggg \lambda^{-1} \gg 1. \tag{9.8.54}$$

This is the first instance in which we have encountered an absolute limitation on T. We note, whoever, that such a limitation was already implied by the conditions (9.8.13), (9.8.27) and (9.8.28):

$$T \gg T\Delta s_{11} \sim 12 \frac{1}{\lambda} \frac{L}{M} \ggg 1 \,. \tag{9.8.55}$$

The zero-point oscillations of the test body are, for measurement theoretical purposes, the same as those computed nonrelativistically in Section 5. This follows at once from a consideration of the relative magnitudes of the elements of the lower right hand corner of the wave-operator matrix appearing in Eq. (9.6.29). In view of the conditions (9.8.13) and (9.8.29) the mass density of the test body is extremely small compared to unity, and therefore $F_{\alpha\epsilon''}$, $F_{\alpha}{}^{\epsilon''\zeta''}$, $F^{\alpha\beta}{}_{\epsilon''}$ are negligible compared to $F^{\alpha\beta\epsilon''\zeta''}$. The Green's functions $G^{\pm\epsilon''}{}_{\gamma'\delta'}$, $G^{\pm}{}_{\epsilon''\zeta''}{}^{\gamma'}$, $G^{\pm}{}_{\epsilon''\zeta''\gamma'\delta'}$ are consequently negligible compared to $G^{\pm\epsilon''\gamma'}$, the spatial components of which are well approximated by Eq. (9.5.48) Thus, although the vacuum fluctuations of the gravitational field in principle contribute to the zero-point oscillations of the test body, in practice they may be neglected.[31] Condition (9.5.59) may therefore be invoked directly to show that the imprecision of the boundary of the test body due to the initial position uncertainties of its component particles is completely negligible. The only question which remains concerns the diffusion of this boundary in time due to the statistical distribution of zero-point velocities which exists just prior to the decoupling of the particles. The magnitude of this diffusion is determined by the average value of the product $v_a v_a$ in the ground state. Making use of Eq. (9.6.69) and neglecting all the propagation functions except $G_{ab'}$, we have, repeating the arguments which led up to Eq. (9.5.53),

$$\langle v_a v_a \rangle = \lim_{x' \to x} i\, G^{(+)}{}_{aa',(0)(0')} \quad . \tag{9.8.56}$$

Introducing the Fourier decomposition (9.5.48) and the phenomenological cut-off (9.5.56), we then find

$$\langle v_a v_a \rangle = \frac{\pi^2}{m\ell}\,(2c_t + c_\ell) \ . \tag{9.8.57}$$

In view of the conditions (9.5.60), (9.5.61) and (9.8.16) it therefore follows that

$$\langle v_a v_a \rangle^{\frac{1}{2}} T \ll L , \tag{9.8.58}$$

which shows that the boundary diffusion may also be neglected.

The motion imparted to the test body in the photon-particle collision process cannot, however, be similarly ignored. It must, in fact, be cancelled. This is accomplished by a procedure due to Bohr and Rosenfeld. Immediately after the time interval Δt, during which the initial strain momentum π' is measured, we give each particle an impulse which is precisely opposite to the impulse it received from the radiation bundle, whereby the particle is again brought to "rest," to the same degree of accuracy as previously permitted by the zero point oscillations. In the present case this is conveniently accomplished by having each particle emit a burst of photons in an appropriate direction. The same process may also be used <u>during</u> the time interval Δt in the transverse slab used for the entrapment and delay of the original photons, in order to prevent uncontrollable displacements from occurring in this region. The stress-energy density of the additional photons---and hence their gravitational effect---can, like that of the original radiation bundle, be taken into account to the accuracy δT given by Eq. (9.8.52).

At the end of the time interval T the same strain-momentum measurement process must be repeated, to obtain a value for π''. Immediately following this measurement, however, the interparticle coupling forces are restored and the test body resumes its previous dimensions.[32] The uncontrollable part of the stress-energy density therefore vanishes prior to the time interval T while inside of this interval it is equal to that produced by a constant uniform strain Δs_{11}. After the interval T a memory of the strain Δs_{11} is left in the contribution which it makes to the vibrational energy upon restoration of the elastic forces. This contribution is, however, of order

$(\Delta s_{11})^2$ and is easily seen to be negligible.[33] The (0)1 and 11 components of he uncertainty in the strain tensor are therefore expressible, to good accuracy, in the forms

$$\Delta T_{(0)(0)} = \{-\rho_0 \Delta s_{11}\, \theta(x^1 + \tfrac{1}{2}L)\, \theta(\tfrac{1}{2}L - x^1) + \tfrac{1}{2}\rho_0 L \Delta s_{11}\, [\delta(x^1 + \tfrac{1}{2}L) + \delta(\tfrac{1}{2}L - x^1)]\} \times \theta(x^2 + \tfrac{1}{2}L)\theta(\tfrac{1}{2}L - x^2)\theta(x^3 + \tfrac{1}{2}L)\theta(\tfrac{1}{2}L - x^3)\theta(x^0)\theta(T - x^0), \quad (9.8.59)$$

$$\Delta T_{(0)1} = \Delta T_{1(0)} = -\rho_0 \Delta s_{11} x^1 \theta(x^1 + \tfrac{1}{2}L)\theta(\tfrac{1}{2}L - x^1)\theta(x^2 + \tfrac{1}{2}L)\theta(\tfrac{1}{2} - x^2) \times \theta(x^3 + \tfrac{1}{2}L)\theta(\tfrac{1}{2}L - x^3)[\delta(x^0) - \delta(T - x^0)], \quad (9.8.60)$$

the use of delta functions to represent effects at the boundary of the space-time averaging domain being permitted because of the smallness of Δs_{11} and Δt. The first term inside the curly bracket in Eq. (9.8.59) represents the effect of the change in mass dnesity due to the expansion Δs_{11} while the second term represents the additional surface layer of mass on the ends of the test body produced by this expansion. Equation (9.8.60) describes the momentum density associated with the sudden changes in the strain at the beginning and end of the interval T. It is readily verified that

$$\Delta T_{(0)}{}^{(0)}{}_{,0} + \Delta T_{(0)}{}^{1}{}_{,1} = 0 , \quad (9.8.61)$$

inconformity with the requirement that the uncertainty in the stress-energy density must be independently conserved. This requirement, in fact, leads us to infer, from the conservation law

$$\Delta T_1{}^{(0)}{}_{,0} + \Delta T_1{}^1{}_{,1} = 0 , \tag{9.8.62}$$

the existence of a momentum flux component

$$\Delta T_{11} = \tfrac{1}{2}\rho_0 \Delta s_{11}(x^1 + \tfrac{1}{2}L)(\tfrac{1}{2}L - x^1)\theta(x^1 + \tfrac{1}{2}L)\theta(\tfrac{1}{2}L - x^1)\theta(x^2 + \tfrac{1}{2}L)\theta(\tfrac{1}{2}L - x^2)$$
$$\times \theta(x^3 + \tfrac{1}{2}L)\theta(\tfrac{1}{2}L - x^3)[\delta'(x^0) + \delta'(T - x^0)] , \tag{9.8.63}$$

describing the uncertainty in the exchange of momentum between the test body and the two radiation bundles. It will be noted that the above conservation laws may be immediately obtained from the compact and (as it turns out) very important representations

$$\left.\begin{aligned} \Delta T_{(0)(0)} &= \tfrac{1}{12} ML^2T \,\Delta s_{11}\, W_{E,11} , \\ \Delta T_{(0)}{}^1 &= \tfrac{1}{12} ML^2T \,\Delta s_{11}\, W_{E,01} , \\ \Delta T_{11} &= \tfrac{1}{12} ML^2T \,\Delta s_{11}\, W_{E,00} , \end{aligned}\right\} \tag{9.8.64}$$

where W_E is the weight function (9.8.19). All the other components of $\Delta T_{\mu\nu}$ vanish. In an arbitrary quasi-Cartesian coordinate system we have

$$\left.\begin{aligned} \Delta T_{(0)(0)} &= \tfrac{1}{12} ML^2T \,\Delta s\, \alpha_a \alpha_b W_{E,ab} , \\ \Delta T_{(0)a} &= \tfrac{1}{12} ML^2T \,\Delta s\, \alpha_a \alpha_b W_{E,0b} , \\ \Delta T_{ab} &= \tfrac{1}{12} ML^2T \,\Delta s\, \alpha_a \alpha_b W_{E,00} , \end{aligned}\right\} \tag{9.8.65}$$

where the α_a are the direction cosines of the x^1-axis fixed in the test body and the subscripts on the strain uncertainty have been dropped.

The gravitational effect of the uncontrollable component of the stress-energy density will be examined in the case of the measurement of a single field average $\bar{\bar{E}}^{I}$. Due to the self-actions of the test body Eq.(9.8.10) must now be replaced by

$$\pi_I'' - \pi_I' = \frac{1}{12} M_I L_I^2 T_I (\bar{\bar{E}}^{I} + \bar{\bar{E}}^{I,I}) . \qquad (9.8.66)$$

Here $\bar{\bar{E}}^{I,I}$ represents the gravitational field due to the entire test-body-photon complex averaged over the space-time domain defined by the test body itself. If this were completely controllable it could be computed in advance and used in the equation

$$\bar{\bar{E}}^{I} = 12\, M_I^{-1}\, L_I^{-2}\, T_I^{-1}\, (\pi_I'' - \pi_I') - \bar{\bar{E}}^{I,I} \qquad (9.8.67)$$

to express the field average $\bar{\bar{E}}^{I}$ which would exist in the absence of the test body in terms of the experimental data π' and π''. As it is, however, the uncertainty in the stress-energy density gives rise to an uncertainty ΔE_{ab} in the field, which satisfies equation (9.7.27) with E_{ab} and $T_{\mu\nu}$ replaced by ΔE_{ab} and $\Delta T_{\mu\nu}$ respectively. Solving this equation with the aid of the retarded Green's function $D^{-}(x - x')$ and making use of Eqs. (9.8.65) we easily find

$$\Delta\bar{\bar{E}}^{I,I} = \frac{1}{12} M_I\, L_I^2\, T_I A^{I,I} \Delta s^I , \qquad (9.8.68)$$

where $A^{I,I}$ is the quantity defined by Eq.(9.8.21), taken with the two space-time domains identical. Making use of the uncertainty relation (9.8.9) we may therefore write the total uncertainty in the measurement of $\bar{\bar{E}}^{I}$ in the form

$$\Delta\bar{\bar{E}}^{I} \sim \frac{12}{M_I L_I^2 T_I \Delta s^I} + \frac{1}{12} M_I L_I^2 T_I |A^{I,I}| \Delta s^I . \qquad (9.8.69)$$

Upon minimization with respect to Δs^I this reduces to

$$\Delta \bar{E}^I \sim |A^{I,I}|^{\frac{1}{2}}, \qquad (9.8.70)$$

which is a special case of Eq. (9.2.30) of Part I.

As was pointed out in Section 2 the measurement of any single observable should be performable with unlimited accuracy, and a method was indicated for accomplishing this by introducing a compensation mechanism. In the present case it is convenient to choose a compensation mechanism which is a generalization of the system of mechanical springs introduced by Bohr and Rosenfeld. Instead of allowing the elastic moduli to fall completely to zero during the time interval T_I we hold the component c_{1111} at the value

$$c_{1111} = \frac{1}{24} M_I{}^2 L_I{}^{-1} T_I (x^1 + \frac{1}{2} L_I)(L_I - \frac{1}{2} x^1) A^{I,I} \quad . \qquad (9.8.71)$$

We note that this value is completely determined by the parameters of the measurement and is therefore known in advance. With nonvanishing c_{1111} the uncontrollable strain Δs^I will give rise to mechanical forces causing additional displacement $\Delta_c \delta z_1$ of the constituent particles of the test body, which are determined by the equation

$$\begin{aligned} \Delta_c \delta \ddot{z}_{1,1} &= -\rho_0{}^{-1} \Delta t_{11,11} = \rho_0{}^{-1} (c_{1111} \Delta s^I)_{,11} \\ &= -\frac{1}{12} \rho_0{}^{-1} M_I{}^2 L_I{}^{-1} T_I A^{I,I} \Delta s^I \, . \qquad (9.8.72) \end{aligned}$$

These displacements make the following contribution to the strain momentum during the time interval T_I:

$$\Delta_c \pi_I = \int_{V_I} \rho_0 x^1 (\Delta_c \delta \dot{z}_1'' - \Delta_c \delta \dot{z}_1') d^3 \underset{\sim}{x}$$

$$= \frac{1}{2}\int_{V_I T_I} \rho_0 (x^1 + \frac{1}{2} L_I)(\frac{1}{2} L_I - x') \Delta_c \delta \ddot{z}_{1,1} \, d^4x$$

$$= -\frac{1}{12} M_I L_I^2 T_I \Delta \bar{\bar{E}}^{I,I} . \qquad (9.8.73)$$

The uncertainty relations following from Eq.(9.8.66) should therefore be modified to read

$$\Delta\pi_I \sim \frac{1}{12} M_I L_I^2 T_I (\Delta\bar{\bar{E}}^I + \Delta\bar{\bar{D}}^{I,I}) + \Delta_c \pi_I$$

$$= \frac{1}{12} M_I L_I^2 T_I \Delta\bar{\bar{E}}^I , \qquad (9.8.74)$$

which is equivalent to (9.8.12), showing that the effects produced by the uncontrollable strain Δs^I have now been cancelled.

It is still necessary to show that although the elastic modulus (9.8.71) produces a significant effect on the strain momentum, its effect on the strain itself during the interval T_I is negligible. This will be the case if the period of oscillation of the strain produced by this modulus is long compared to T_I. From Eq. (9.8.72) we see that this period is given by

$$T_c = (\frac{1}{12} \rho_0^{-1} M_I^2 L_I^{-1} T_I |A^{I,I}|)^{-\frac{1}{2}} . \qquad (9.8.75)$$

Remembering that $|A^{I,I}| \sim (R_{crit})^2$ and making use of (9.8.13) and (9.8.23), we have

$$\frac{T_c}{T_I} \sim \frac{L_I}{T_I} (12 \frac{L_I}{M_I})^{\frac{1}{2}} \gg 1 , \qquad (9.8.76)$$

which establishes the utility of the compensation mechanism.[34]

We come finally to the verification of the uncertainty relation (9.5.22) for the measurement of two field averages $\bar{\bar{E}}^I$ and $\bar{\bar{E}}^{II}$. Equation (9.8.66) must be replaced by the pair of equations

$$\left.\begin{aligned} \pi_I'' - \pi_I' &= \frac{1}{12} M_I L_I{}^2 T_I(\bar{\bar{E}}^I + \bar{\bar{E}}^{I,I} + \bar{\bar{E}}^{II,I}) \ , \\ \pi_{II}'' - \pi_{II}' &= \frac{1}{12} M_{II} L_{II}{}^2 T_{II}(\bar{\bar{E}}^{II} + \bar{\bar{E}}^{II,II} + \bar{\bar{E}}^{I,II}) \ , \end{aligned}\right\} \qquad (9.8.77)$$

where the notation is obvious. Here the mutual uncertainties in the measurements cannot be completely cancelled. As has been pointed out in Section 2, the greatest possible mutual accuracy requires the use not only of compensation mechanisms but also of a correlation mechanism. We consider first the case in which the space-time domains $V_I T_I$ and $V_{II} T_{II}$ overlap. The appropriate non-compensations and correlations are in this case achieved by introducing non-vanishing elastic moduli c_{1111} in the two test bodies during the respective time intervals T_I and T_{II}, and bringing into action, during the interval of overlap of T_I and T_{II}, a set of mechanical springs linking adjacent component particles of the two bodies in the region of spatial overlap. Except in the case in which the x^1-axes of the two bodies are parallel the linkage between the bodies should not be direct; in the general case the elastic coupling forces should be transmitted through a set of bent levers, similar to the one shown in Fig (9-4), the pivots of which are fastened to a third elastic body which freely interpenetrates the other two, has comparable mass, and remains stiff throughout the entire measurement process. It is not hard to show that Eqs. (9.8.77) then take the modified forms

$$\left.\begin{aligned} \pi_I'' - \pi_I' &= \frac{1}{12} M_I L_I{}^2 T_I(\bar{\bar{E}}^I + \bar{\bar{E}}^{I,I} + \bar{\bar{E}}^{II,I}) - \kappa_I s^I - \kappa_{I,II} s^{II} \ , \\ \pi_{II}'' - \pi_{II}' &= \frac{1}{12} M_{II} L_{II}{}^2 T_{II}(\bar{\bar{E}}^{II} + \bar{\bar{E}}^{II,II} + \bar{\bar{E}}^{I,II}) - \kappa_{II} s^{II} - \kappa_{I,II} s^I \ , \end{aligned}\right\} \qquad (9.8.78)$$

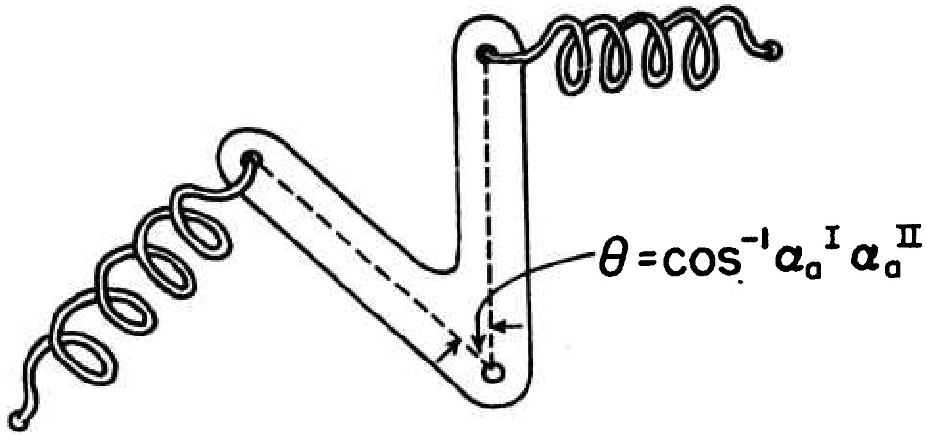

Fig. 9-4. Correlation device.

where the coefficients κ_I , κ_{II} , $\kappa_{I,II}$ are independent linear functions of the (essentially three) independent elastic constants involved in the compensation and correlation devices (see Yeh, 1960). Taking note of the uncertainty relations $\Delta\pi_I \Delta s^I \sim 1$, $\Delta\pi_{II}\Delta s^{II} \sim 1$, and the fact that

$$\Delta\bar{\bar{E}}^{I,II} = \frac{1}{12} M_I L_I{}^2 T_I A^{I,II} \Delta s^I \text{ , etc. ,} \tag{9.8.79}$$

we see that by choosing the elastic constants in such a way that

$$\left.\begin{aligned} \kappa_I &= \frac{1}{144} \quad M_I{}^2 L_I^4 \; T_I{}^2 A^{I,I} \quad , \\ \kappa_{II} &= \frac{1}{144} \quad M_{II}{}^2 L_{II}{}^4 T_{II}{}^2 A^{II,II} \quad , \\ \kappa_{I,II} &= \frac{1}{288} \quad M_I M_{II} L_I{}^2 L_{II}{}^2 T_I T_{II} (A^{I,II} + A^{II,I}) \quad , \end{aligned}\right\} \tag{9.8.80}$$

we may reduce the uncertainty relations following from Eqs. (9.8.78) to the forms

$$\Delta\bar{\bar{E}}^I \sim \frac{12}{M_I L_I{}^2 T_I \Delta s^I} + \frac{1}{24} M_{II} L_{II}{}^2 T_{II} | A^{I,II} - A^{II,I} | \Delta s^{II} , \tag{9.8.81}$$

$$\Delta\bar{\bar{E}}^{II} \sim \frac{12}{M_{II} L_{II}{}^2 T_{II} \Delta s^{II}} + \frac{1}{24} M_I L_I{}^2 T_I | A^{I,II} - A^{II,I} | \Delta s^I , \tag{9.8.82}$$

the product of which, upon minimization with respect to the product $\Delta s^I \Delta s^{II}$, reduces to (9.8.22).

The case in which the time intervals T_I and T_{II} overlap while the spatial regions do not has been treated for the electromagnetic measurement problem by Bohr and Rosenfeld. Their method may be immediately applied also to the gravitational problem, but we refer the reader to their paper for details. Evidently there will be no mutual interference between the two measurements at all in this case unless the test bodies are separated by distances of order T or less. Since we have always assumed $L > T$, the case of spatial overlap and the case in which the time intervals do not overlap are really sufficient to provide an adequate test of the quantum formalism. In the latter case the introduction of a correlation mechanism is actually unnecessary since either $A^{I,II}$ or $A^{II,I}$ (or both) vanishes. Only the individual compensation mechanisms are then needed. It should, of course, always be remembered that the explicit form (9.8.21) for the quantities $A^{I,II}$, etc. is valid only in local regions. If the two test bodies are situated at large distances from one another, they will continue to provide a valid test of the quantum formalism, but the Green's function $D^{-}(x - x')$ appearing in the uncertainty relation will have to be replaced by the function $G^{-}{}_{\mu\nu\sigma'\tau'}$ of Eq. (9.6.55), which takes into account the effect of the macroscopic curvature of space-time on the propagation of disturbances in the gravitational field.

We conclude this section with an outline of the measurability analysis for the components H_{ab} of the Riemann tensor. The test body which is appropriate for measuring averages of these components consists of two mutually interpenetrable cubes having identical masses. Each is composed of a uniform distribution of spinning particles, the spins of one being aligned antiparallel to those of the other, so that each has a uniform mass density ρ_0 and a uniform spin density, the latter being denoted by σ_a for one and $-\sigma_a$ for the other.

The measurement process is based on the ponderomotive equation (9.7.35) for spinning particles. Outside of the time interval T the two cubes are bound firmly together and possess elastic moduli which render them stiff.

During the interval T the elastic moduli drop to zero, the mutual binding is released, and the cubes are allowed to drift apart. The mutual separation δz_a at any point satisfies the equation

$$\rho_0 \delta \ddot{z}_a = 2\sigma_b H_{ba} , \qquad (9.8.83)$$

the factor 2 reflecting the fact that each cube possesses a spin angular momentum. The measurement of an average of the compnent H_{ba} may evidently be achieved by orienting the spins in the positive and negative x^b-directions and making, through a symmetric exchange of radiation bundles between the two cubes, measurements of the total relative momenta of the two cubes in the x^a-direction at the beginning and end of the interval T. More generally the spins may be oriented along an arbitrary axis characterized by direction cosines $\pm\beta_a$. Writing

$$\sigma_a = \sigma\beta_a , \qquad (9.8.84)$$

and taking note of the fact that the "reduced mass" characterizing the relative momenta corresponds to a density $\frac{1}{2}\rho_0$, we then obtain, as a result of measurements of the relative momenta in the x^1-direction,

$$p_1'' - p_1' = \frac{1}{2}\int_V \rho_0(\delta\dot{z}_1'' - \delta\dot{z}_1')d^3\underset{\sim}{x} = \frac{1}{2}\int_{VT} \rho_0 \delta\ddot{z}_1 \, d^4x = \Sigma T \beta_b \bar{\bar{H}}_{b1} , \qquad (9.8.85)$$

$$\bar{\bar{H}}_{ab} \equiv L^{-3} T^{-1} \int_{VT} H_{ab} \, d^4x , \qquad (9.8.86)$$

$$\Sigma \equiv L^3\sigma . \qquad (9.8.87)$$

The weight factor for the field average is seen to be uniform in this case instead of parabolic.

We shall not give here a detailed description of the radiation bundles used for the relative momentum measurements, nor shall we repeat the analysis of the various sources of uncertainty which arise in the measurement process. It is clear that the arguments are essentially the same for $\beta_b\bar{\bar{H}}_{b1}$ as they were for $\bar{\bar{E}}_{11}$, the main difference being that the photon distribution at the moment of collision is now uniform instead of proportional to $|x^1|$. The only source of uncertainty which, in the end, must be taken into account is that due to the uncontrollable relative displacement Δx^1 resulting from the first relative momentum measurement. This relative displacement is, of course, held essentially constant throughout the time interval T through the use of a "counter impulse" as in the case of the measurement of $\bar{\bar{E}}$, and is related to the uncertainty Δp_1 in the relative momentum measurements by

$$\Delta x^1 \Delta p_1 \sim 1 \quad . \tag{9.8.88}$$

The uncertainty in the measurement of $\beta_b\bar{\bar{H}}_{b1}$ is therefore

$$\Delta(\beta_b H_{b1}) \sim \frac{1}{\Sigma T \Delta x^1} \quad , \tag{9.8.89}$$

which, for every value of Δx^1 no matter how small, can be made arbitrarily small by the choice of a sufficiently large value of Σ. The critical field strength below which we enter the quantum domain is given by Eq. (9.8.23) as before, and the condition

$$\Delta(\beta_b\bar{\bar{H}}_{b1}) \sim \lambda L^{-3} \quad , \qquad \pi \ll 1 \quad , \tag{9.8.90}$$

on the accuracy of the measurement, together with the necessary restriction

$$\Delta x^1 \ll L \, , \tag{9.8.91}$$

yields the requirement

$$\Sigma \sim \frac{1}{\lambda} \frac{L}{\Delta x^1} \frac{L}{T} L \ggg L. \tag{9.8.92}$$

In the present analysis the total relative spin 2Σ behaves like a charge, and because it is independent of the total mass $2M$ we have here an adjustable charge-to-mass ratio Σ/M. The mass must be chosen large enough so that the relative displacement of the two cubes caused by the field to be measured remains small compared to L during the interval T. The condition for this is

$$(\Sigma/M)\beta_b H_{b1} T^2 \ll L \quad . \tag{9.8.93}$$

Remembering the weak field condition

$$\beta_b H_{b1} T^2 \ll 1 \quad , \tag{9.8.94}$$

[cf. (9.8.15)] we see that this will be satisfied if we choose

$$M \gtrsim \Sigma L^{-1} \ggg 1 \tag{9.8.95}$$

[cf. (9.8.28)], which, together with the condition (9.8.13) leads once again to the fundamental limitation (9.8.29) on the smallness of allowable measurement domains. We note that the effect of the spins themselves on the gravitational field may be ignored except during the interval T, since at other times they cancel one another. We also note that conditions (9.8.92) and (9.8.29) together imply that a very large number of elementary spins ($\frac{1}{2}$, 1, $\frac{3}{2}$, etc.) must be used in the construction of the test body.

The commutation relations which remain to be tested are

$$[\bar{\bar{E}}^{I}, \bar{\bar{H}}^{II}] = \frac{1}{4} i \int d^4x \int d^4x' \, \alpha_a{}^I \alpha_b{}^I \beta_c{}^{II} \alpha_d{}^{II} W_E{}^I(x) W_H{}^{II}(x')$$

$$\times \epsilon_{cef}(\delta^T{}_{ae}\delta^T{}_{bd} + \delta^T{}_{ad}\delta^T{}_{be} - \delta^T{}_{ab}\delta^T{}_{ed})\nabla^2 D_{,f}(x - x'), \qquad (9.8.96)$$

$$[\bar{\bar{H}}^{I}, \bar{\bar{H}}^{II}] = \frac{1}{4} i \int d^4x \int d^4x' \beta_a{}^I \alpha_b{}^I \beta_c{}^{II} \alpha_d{}^{II} W_H{}^I(x) W_H{}^{II}(x')$$

$$\times(\delta^T{}_{ac}\delta^T{}_{bd} + \delta^T{}_{ad}\delta^T{}_{bc} - \delta^T{}_{ab}\delta^T{}_{cd})\nabla^4 D(x - x'), \qquad (9.8.97)$$

where the subscripts on the $\bar{\bar{E}}$ and $\bar{\bar{H}}$ have been dropped and an arbitrary quasi-Cartesian coordinate system has been introduced relative to which the spins and x^1-axes of the test bodies have direction cosines $\beta_a{}^I$, $\beta_a{}^{II}$ and $\alpha_a{}^I$, $\alpha_a{}^{II}$ respectively. The function $W_H(x)$ is the weight function appropriate to the measurement of $\bar{\bar{H}}$, having, in a coordinate system oriented with the test body, the form

$$W_H(x) \equiv L^{-3}T^{-1}\theta(x^1 + \tfrac{1}{2} L)\theta(\tfrac{1}{2} L - x^1)\theta(x^2 + \tfrac{1}{2} L)\theta(\tfrac{1}{2} L - x^2)$$

$$\times \theta(x^3 + \tfrac{1}{2} L)\theta(\tfrac{1}{2} L - x^3) \, \theta(x^0)\theta(T - x^0). \qquad (9.8.98)$$

By the same procedure as was used to obtain Eq. (9.8.20) we may re-express the commutators (9.8.96) and (9.8.97) in the forms

$$[\bar{\bar{E}}^{I}, \bar{\bar{H}}^{II}] = i\,(C^{I,II} - D^{II,I}) \,, \qquad (9.8.99)$$

$$[\bar{\bar{H}}^{I}, \bar{\bar{H}}^{II}] = i\,(B^{I,II} - B^{II,I}) \,, \qquad (9.8.100)$$

where

$$B^{I,II} \equiv \frac{1}{4}\int d^4x \int d^4x' \, W_H{}^{II}(x') D^-(x' - x)\beta_a{}^I \alpha_b{}^I \beta_c{}^{II} \alpha_\alpha{}^{II}$$

$$\times[(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd})W_H{}^I{}_{,0000}(x)$$

$$- \delta_{ac}W_H{}^I{}_{,bd00}(x) - \delta_{ad}W_H{}^I{}_{,bc00}(x) - \delta_{bc}W_H{}^I{}_{,ad00}(x) - \delta_{bd}W_H{}^I{}_{,ac00}(x)$$

$$+ \delta_{ab}W_H{}^I{}_{,cd00}(x) + \delta_{cd}W_H{}^I{}_{,ab00}(x) + W_H{}^I{}_{,abcd}(x)] \,, \qquad (9.8.101)$$

$$C^{I,II} \equiv \frac{1}{4}\int d^4x \int d^4x' \, W_H{}^{II}(x') \, D^-(x' - x) \, \alpha_a{}^{I}\alpha_b{}^{I}\beta_c{}^{II}\alpha_d{}^{II}\epsilon_{cef}$$

$$\times [(2\,\delta_{ae}\delta_{bd} - \delta_{ab}\delta_{ed})W_E{}^{I}{}_{,000f}(x)$$

$$- 2\,\delta_{ae}W_E{}^{I}{}_{,bd0f}(x) + \delta_{ed}W_E{}^{I}{}_{,ab0f}(x)] \, , \qquad (9.8.102)$$

$$D^{II,I} \equiv \frac{1}{4}\int d^4x \int d^4x' \, W_E{}^{I}(x) \, D^-(x - x') \, \alpha_a{}^{I}\alpha_b{}^{I}\beta_c{}^{II}\alpha_d{}^{II}\epsilon_{cef}$$

$$\times [(2\,\delta_{ae}\delta_{bd} - \delta_{ab}\delta_{ed})W_H{}^{II}{}_{,000f}(x')$$

$$- 2\,\delta_{ae}W_H{}^{II}{}_{,bd0f}(x') + \delta_{ed}W_H{}^{II}{}_{,ab0f}(x')] \, . \qquad (9.8.103)$$

We consider first the commutator (9.8.99), the testing of which requires test bodies appropriate to the measurement of both types of components, $\bar{\bar{E}}$ and $\bar{\bar{H}}$, of the Riemann tensor, together with suitable compensation and correlation mechanisms. A compensation mechanism suitable for the measurement of $\bar{\bar{E}}$ has already been described. The compensation mechanism which may conveniently be used in the test body which measures $\bar{\bar{H}}$ consists of a set of mechanical springs joining the two interpenetrating cubes of which the test body is composed. The correlation mechanism may consist, as before, of a set of springs in the region of spatial overlap, connecting the two test bodies through pivoting devices as shown in Fig. (9-4). In this case it is important, however, that the springs be affixed to only one of the two interpenetrating cubes; otherwise no correlation is achieved. Under an arrangement of this type the dynamical equations describing the measurement process take the forms

$$\left.\begin{aligned} \pi_I{}'' - \pi_I{}' &= \frac{1}{12} M_I L_I{}^2 T_I(\bar{\bar{E}}^{I} + \bar{\bar{E}}^{I,I} + \bar{\bar{E}}^{II,I}) - \kappa_I s^I - \kappa_{I,II} x^{II} \, , \\ p_{II}{}'' - p_{II}{}' &= \Sigma_{II} T_{II}(\bar{\bar{H}}^{II} + \bar{\bar{H}}^{II,II} + \bar{\bar{H}}^{I,II}) - \kappa_{II} x^{II} - \kappa_{I,II} s^I \, . \end{aligned}\right\} \qquad (9.8.104)$$

The quantity $\bar{\bar{H}}^{I,II}$ represents the contribution which test body I makes to the field average $\bar{\bar{H}}$ over the space-time domain defined by test body II . Its uncertainty may be computed by inserting the stress-energy uncertainty (9.8.65) into equation (9.7.29) (with H_{ab} and $T_{\mu\nu}$ replaced by ΔH_{ab} and $\Delta T_{\mu\nu}$ respectively) and solving the latter with the aid of the retarded Green's function $D^{-}(x - x')$. One finds

$$\Delta\bar{\bar{H}}^{I,II} = \frac{1}{12} M_I L_I{}^2 T_I C^{I,II} \Delta s^I . \tag{9.8.105}$$

In order to compute in a similar manner the uncertainty $\Delta\bar{\bar{E}}^{II,I}$ in the contribution which test body II makes to the field average $\bar{\bar{E}}$ over the space-time domain defined by test body I , we must first determine the form of the stress-energy uncertainty of the test body II resulting from the measurement of the relative momentum $p_{II}{}'$. This uncertainty receives contributions from two sources: from the mass and from the spin of the test body. At first sight it would appear that the mass contributes a dipole term to $\Delta T_{(0)(0)}$ which is proportional to Δx^{II} . It is to be remembered, however, that Δx^{II} is the <u>relative</u> displacement of the two cubes composing the test body. The center of mass of the test body as a whole remains at rest in the coordinate system originally defined by the body.[35] Therefore the mass contribution is that of two equal and outwardly oriented dipoles at opposite ends of the test body. Furthermore the strength of these dipoles is proportional to $(\Delta x^{II})^2$ and <u>not</u> to Δx^{II} . The mass contribution may therefore be neglected in comparison to the spin contribution which is of the first order in Δx^{II} . It is not hard to see that the latter contribution, in a coordinate system oriented with the test body, is given by

$$\Delta T_{(0)(0)} = 0 \ , \tag{9.8.106}$$

$$\Delta T_{a(0)} = -\frac{1}{2}\epsilon_{abc}\beta_c\sigma\Delta x^1\{[\delta(\frac{1}{2}L - x^1) - \delta(x^1 + \frac{1}{2}L)]\theta(x^2 + \frac{1}{2}L)\theta(\frac{1}{2}L - x^2) \times \theta(x^3 + \frac{1}{2}L)\theta(\frac{1}{2}L - x^3)\}_{,b}\theta(x^0)\theta(T - x^0) \ , \tag{9.8.107}$$

$$\Delta T_{ab} = \frac{1}{2}(\delta_a{}^1\epsilon_{bcd} + \delta_b{}^1\epsilon_{acd})\beta_d\sigma\Delta x^1\{\theta(x^1 + \frac{1}{2}L)\theta(\frac{1}{2}L - x^1) \times \theta(x^2 + \frac{1}{2}L)\theta(\frac{1}{2}L - x^2)\theta(x^3 + \frac{1}{2}L)\theta(\frac{1}{2}L - x^3)\}_{,c}[\delta(x^0) - \delta(T - x^0)], \tag{9.8.108}$$

where the label "II" has been temporarily omitted and where the implicit assumption has been made that all elastic and binding forces are restored at the end of the interval T so that the test body returns to its original state (except for slight changes resulting from its intervening experiences which may be neglected). Equation (9.8.107) is obtained immediately from Eq.(D.15) Appendix D by carrying out a differentiation with respect to x^1 to represent the effect of the displacement Δx^1. The components ΔT_{ab} given by Eq. (9.8.108) are then inferred from the energy-momentum conservation laws. One easily verfies that

$$\Delta T_{(0)}{}^{(0)}{}_{,0} + \Delta T_{(0)}{}^{a}{}_{,a} = 0 \ , \tag{9.8.109}$$

$$\Delta T_a{}^{(0)}{}_{,0} + \Delta T_a{}^{b}{}_{,b} = 0 \ . \tag{9.8.110}$$

These relations also follow at once from the representation

$$\left.\begin{aligned} \Delta T_{(0)(0)} &= 0 \ , \\ \Delta T_{a(0)} &= \frac{1}{2}\Sigma T\Delta x^1\epsilon_{abc}\beta_c W_{H,1b} \ , \\ \Delta T_{ab} &= \frac{1}{2}\Sigma T\Delta x^1(\delta_a{}^1\epsilon_{bcd} + \delta_b{}^1\epsilon_{acd})\beta_d W_{H,0c} \ , \end{aligned}\right\} \tag{9.8.111}$$

which, in an arbitrary quasi-Cartesian coordinate system, takes the form

$$\left.\begin{aligned}\Delta T_{(0)(0)} &= 0 \ , \\ \Delta T_{a(0)} &= \frac{1}{2}\Sigma T\Delta x\beta_c\alpha_d\epsilon_{abc}W_{H,bd} \ , \\ \Delta T_{ab} &= \frac{1}{2}\Sigma T\Delta x\beta_c\alpha_d(\delta_{ad}\epsilon_{bec} + \delta_{bd}\epsilon_{aec})W_{H,0e} \ .\end{aligned}\right\} \qquad (9.8.112)$$

Inserting (9.8.112) into the retarded solution of Eq.(9.7.27), we finally get

$$\Delta\bar{\bar{E}}^{II,I} = \Sigma_{II}T_{II}D^{II,I}\Delta x^{II} \ . \qquad (9.8.113)$$

Similarly, inserting (9.8.112) into the retarded solution of Eq. (9.7.28) and making use of the identity

$$\begin{aligned}\epsilon_{abc}\epsilon_{def} \equiv\ & \delta_{ad}\delta_{be}\delta_{cf} + \delta_{ae}\delta_{bf}\delta_{cd} + \delta_{af}\delta_{bd}\delta_{ce} \\ & - \delta_{ad}\delta_{bf}\delta_{ce} - \delta_{af}\delta_{be}\delta_{cd} - \delta_{ae}\delta_{bd}\delta_{cf} \ ,\end{aligned} \qquad (9.8.114)$$

we find

$$\Delta\bar{\bar{H}}^{II,II} = \Sigma_{II}T_{II}B^{II,II}\Delta x^{II} \ . \qquad (9.8.115)$$

The uncertainty $\Delta\bar{\bar{E}}^{I,I}$ is given by Eq.(9.8.68) as before.

If we now choose the various elastic constants in such a way that

$$\left.\begin{aligned}\kappa_I &= \frac{1}{144} M_I{}^2L_I{}^4T_I{}^2A^{I,I} \ , \\ \kappa_{II} &= \Sigma_{II}{}^2T_{II}{}^2B^{II,II} \ , \\ \kappa_{I,II} &= \frac{1}{24} M_I\Sigma_{II}L_I{}^2T_IT_{II}(C^{I,II} + D^{II,I}) \ ,\end{aligned}\right\} \qquad (9.8.116)$$

then the uncertainty relations following from Eqs. (9.8.104) take the forms

$$\Delta\bar{\bar{E}}^{I} \sim \frac{12}{M_I L_I{}^2 T_I \Delta s^I} + \frac{1}{2}\Sigma_{II}T_{II}|C^{I,II} - D^{II,I}|\Delta x^{II} \ , \qquad (9.8.117)$$

$$\Delta\bar{\bar{H}}^{II} \sim \frac{1}{\Sigma_{II}T_{II}\Delta x^{II}} + \frac{1}{24}M_I L_I{}^2 T_I|C^{I,II} - D^{II,I}|\Delta s^{I} \ , \qquad (9.8.118)$$

the product of which, upon minimization with respect to the product $\Delta s^I \Delta x^{II}$, reduces to

$$\Delta\bar{\bar{E}}^{I}\Delta\bar{\bar{H}}^{II} \sim |C^{I,II} - D^{II,I}| \ , \qquad (9.8.119)$$

in accord with the commutation relation (9.8.99).

The testing of the commutation relation (9.8.100) is again entirely analogous to the above. Two pairs of spin-endowed cubes are needed in this case. The compensation and correlation mechanisms again consist of mechanical springs. The springs for the correlation mechanism join a cube from one of the two test bodies to a cube from the other through the usual pivot devices. The dynamical equations describing the measurement process are

$$\left.\begin{aligned} p_I'' - p_I' &= \Sigma_I T_I(\bar{\bar{H}}^{I} + \bar{\bar{H}}^{I,I} + \bar{\bar{H}}^{II,I}) - \kappa_I x^I - \kappa_{I,II}x^{II} \ , \\ p_{II}'' - p_{II}' &= \Sigma_{II} T_{II}(\bar{\bar{H}}^{II} + \bar{\bar{H}}^{II,II} + \bar{\bar{H}}^{I,II}) - \kappa_{II} x^{II} - \kappa_{I,II}x^{I} \ , \end{aligned}\right\} \qquad (9.8.120)$$

The various uncertainties are given by

$$\Delta\bar{\bar{H}}^{I,II} = \Sigma_I T_I B^{I,II}\Delta x^I \ , \text{ etc.} \qquad (9.8.121)$$

Therefore, by choosing the elastic constants in such a way that

$$\left.\begin{aligned}
\kappa_{I} &= \Sigma_{I}{}^{2}T_{I}{}^{2}B^{I,I} \,, \\
\kappa_{II} &= \Sigma_{II}{}^{2}T_{II}{}^{2}B^{II,II} \,, \\
\kappa_{I,II} &= \tfrac{1}{2}\,\Sigma_{I}\Sigma_{II}T_{I}T_{II}(B^{I,II} + B^{II,I}) \,,
\end{aligned}\right\} \qquad (9.8.122)$$

we obtain the uncertainty relations

$$\Delta\bar{\bar{H}}^{I} \sim \frac{1}{\Sigma_{I}T_{I}\Delta x^{I}} + \frac{1}{2}\,\Sigma_{II}T_{II}|B^{I,II} - B^{II,I}|\Delta x^{II} \quad , \qquad (9.8.123)$$

$$\Delta\bar{\bar{H}}^{II} \sim \frac{1}{\Sigma_{II}T_{II}\Delta x^{II}} + \frac{1}{2}\,\Sigma_{I}T_{I}|B^{I,II} - B^{II,I}|\Delta x^{I} \quad , \qquad (9.8.124)$$

the product of which, upon minimization with respect to the product $\Delta x^{I}\Delta x^{II}$, reduces to

$$\Delta\bar{\bar{H}}^{I}\,\Delta\bar{\bar{H}}^{II} \sim |B^{I,II} - B^{II,I}| \,. \qquad (9.8.125)$$

The measurement theoretical verification of the formalism of the quantum theory of geometry to lowest order of perturbation theory (weak-field approximation) is thus complete.

(9.9) Conclusions and outlook.

The conclusions which may be drawn from the investigations of this chapter are the following:

(1) The Uncertainty Principle and the theory of measurement lead to a well defined framework within which to develop a manifestly covariant formalism for the quantum theory of gravitation. The elements which enter naturally into this formalism are the Green's functions describing the propagation of small disturbances. The quantitative results of the formalism are completely unambiguous at the level of the weak-field approximation. Ambiguities connected with the choice of factor sequences can arise, if at all, only in higher orders of perturbation expansions based on the weak-field approximation as a starting point. It should be emphasized once again that the weak-field approximation is an assumption only about the magnitude of the Riemann tensor in any finite domain of interest. It is not an assumption about asymptotic conditions or about the global structure of space-time.

(2) Averages of the gravitational field (i.e., Riemann tensor) over space-time domains having dimensions large compared to 10^{-32} cm. can be measured with a degree of accuracy well within the domain of quantum phenomena provided that test bodies of sufficient refinement but violating no fundamental principles are used. Examination of the mutual interference of such measurements verifies in detail the statistical predictions of the quantum formalism.

(3) The gravitational field, like all other fields, therefore must be quantized, or else the logical structure of quantum field theory must be profoundly altered, or both. The possiblity is left open that the quantum theory of geometry may itself contribute deeply to the future development of quantum field theory.

(4) The dimension 10^{-32} cm. constitutes a fundamental limit on the smallness of allowable measurement domains. Below this limit it is impossible to interpret the results of measurements in terms of properties or states characterizing the individual systems under observation. Although this conclusion was reached within the framework of an investigation based on the weak-field approximation it is obviously of general validity. The very uncertainty in the energy of the devices (e.g., photons) needed to make an observation in such a small region, will in virtue of the uncontrollable gravitational disturbance which it produces, completely destroy the statistical significance of the results of the observation. This is true for the measurement of any field, not only the gravitational field. The concept of "field strength" therefore has, below 10^{-32} cm., no objective meaning in terms of observations performed at the classical level. That is to say, 10^{-32} cm. constitutes an absolute limit on the domain of applicability of classical concepts, even as modified by the Principle of Complementarity.

These conclusions give rise immediately to the following questions: Why does experiment appear to show that a practical limit on the domain of applicability of classical concepts already exists at or near 10^{-13} cm.? Can gravitation, in virtue of this fact, really have any connection with elementary particle physics?

Certainly, doubts must arise when one notes that if test bodies suitable for detecting the quantum properties of the gravitational field are to be constructed out of normal matter, condition (9.8.28) implies that they will will be visible to the naked eye! Even if one could imagine them to be constructed out of nuclear matter their dimensions would have to be at least of the order of a micron. Conversely it is only to bodies of such size that gravitational effects themselves can properly be ascribed. For, if an attempt were made to measure the gravitational field (i.e., Riemann tensor) of such a body, the measurement, like all field measurements, would have to be performed over some finite domain of dimension L, so that even if the mass m of the body were concentrated practically in a point the strongest field which could be measured would be of order $m\,L^{-3}$. But the quantum fluctuations themselves are of order L^{-3} [cf. Eq. (9.8.23)]. Hence, it makes no sense at all to talk about the gravitational field of an individual elementary particle ($m \sim 10^{-20}$ in dimensionless units). The static field exceeds the quantum fluctuations in magnitude only for bodies more massive than 3.07×10^{-6} gram (the unit of mass in the dimensionless system). From this point of view the gravitational field is plainly a statistical phenomenon of bulk matter, although its fluctuations are governed by quantum laws.

Any attempt to bridge the gap between 10^{-32} cm. and 10^{-13} cm. by means of gravitation alone seems practically hopeless. At the very least such an attempt would have to invoke exceedingly complicated processes. Misner and Wheeler (1957) have made a preliminary study of the dynamics and properties of "wormholes" and have suggested that such objects may provide an avenue for connecting gravitation with elementary particles. In view of the fact that a "wormhole" is strictly a classical entity, however, the suggestion must be viewed with a measure of skepticism. It is far from clear that "wormhole" concepts would provide useful mental images in the ultra-microscopic domain except in a purely topological sense. The possible dynamical existence of "wormholes" depends crucially on the nonlinearities of Einstein's equations, but the effect of these nonlinearities must be described in c-number terms. The situation is similar to that which exists in the relation of the theory of exact classical solutions of Einstein's equations to the quantum theory. The case which is sometimes made for discovering and studying the properties of such solutions---particularly the properties of wave solutions---because of their supposed significance for the quantization program, is largely spurious. The study of nonlinearities at the classical level in which large numbers of coherent quanta are involved has little relevance for the description of graviton-graviton interactions. The same is true of other field theories. In electrodynamics, for example, a study of the solutions of the dynamical equations of a charged classical boson field interacting with a classical Maxwell field will never lead to the concept of vacuum polarization, no matter how exhaustively pursued. This does not mean, of course, that classical nonlinear problems are unimportant. Indeed, they arise in the course of fundamental investigations on the behavior of large disturbances. Quantum mechanics, however, is a theory of small disturbances, and the nonlinear problems which arise within its framework are usually abstract and without classical models.

Only in the domain below 10^{-32} cm. is quantum mechanics itself transformed into a theory of large disturbances and violent fluctuations. "Theory," of course, is hardly the proper word to use here since it does not yet exist. Quantum mechanics is certain to be very different from what we know it in this mysterious region. But this brings us back again to the problem of the great gap between 10^{-13} cm. and 10^{-32} cm. It is necessary to admit that something "happens" at 10^{-13} cm. which has every appearance of being fundamental and not merely a statistical manifestation of basic phenomena occurring at a much deeper level. Complexities are present, to be sure, but they do not compare with the complexities of atomic phenomena; and the gap between 10^{-8} cm. and 10^{-13} cm. is negligible compared to that between 10^{-13} cm. and 10^{-32} cm. Since the theory of gravitation has nothing special to say at 10^{-13} cm. it is necessary to look elsewhere for the description of Nature at this level. However, the geometrical viewpoint of general relativity need not be abandoned. In fact, it has never been completely abandoned, as is evidenced from the many attempts to bring order into the description of elementary particle phenomena by introducing "internal" spaces and invariance groups. On the other hand, it has not, since Einstein, been pursued with the single mindedness and integrity which it perhaps deserves. It may, for example, be worth while to make strong attempts to link the apparent "internal" spaces more directly to the ordinary four-dimensional space-time of everyday experience, even at the risk of resurrecting some long abandoned so-called "unified field theories" in modified or generalized form. The present unattractiveness of theories of this type is due at least in part to the lack of a quantum formalism for them. If the quantization program for gravitation can be successfully pushed through then these theories may become more attractive.

The existence of 10^{-13} cm. (or even 10^{-14}cm. or 10^{-15}cm.) as a practical limit on the smallness of measurement domains does not mean that the terminology of field theory ("field strengths,""quanta," "fluctuations," etc.) should be abandoned below this level. Although the concepts embodied in the terminology become, in this domain, purely abstract rather than experimental, no question of "hidden variables" is involved. The continued use of continuous parameters (i.e., coordinates) to describe dynamical systems at this level is an unavoidable requirement of the theory of group representations. The fact that the invariance groups of physics are continuous is established already at the classical level. Except in the case of Abelian groups, continuous groups cannot be successively approximated by finite groups. There is no in-between.[36] Even at 10^{-32}cm. the continuum description must persist if the general coordinate transformation group is really fundamental. Here, however, the use of a phenomenological "cut-off", reflecting the absolute meaninglessness of concepts like "field strength" at this level, may be valid. Deser (1957) has given heuristic arguments for this possibility, based on the Feynman quantization method.[37] Such a "cut-off" would, of course, eliminate the ultra-violet divergences of field theory and establish a fundamental role for gravitation in elementary particle physics. Moreover, the existence of a "cut-off" at this wavelength is not obviously incompatible with the success of modern field theory in correlating experimental data.

A brief look should be taken at the possible form which the quantum theory of geometry may assume in its eventual development. Although it has been shown that the requirement of asymptotic flatness at infinity is not essential to the quantization program it will nevertheless often be a convenient assumption in practice. When asymptotic flatness holds, the linearized theory should provide an excellent framework within which to describe conditions in the remote past and future, when the fields associated with the small number of real quanta involved in any given quantum problem are dispersed to a state of infinite weakness. The Riemann tensor is then effectively a true invariant, and its positive and negative frequency components should be directly usable for the annihilation and creation of initial and final gravitons. The actual interactions between these gravitons as well as the interactions between gravitons and other quanta will then be described in terms of Green's functions. Instead of retarded and advanced Green's functions the Feynman propagator will be appropriate for this description. (It differs from the former only in the nature of its boundary conditions; it satisfies the same basic equations [Eqs. (9.3.7a,b)].) The development, however, should not be kept within the confines of the flat space-time approach. For, an examination of the inevitable infinities of the theory from the Lorentz invariant standpoint leads to a very pessimistic view of the renormalization picture. In opposition to this view it must be constantly borne in mind that the "bad" divergences of quantum gravidynamics are of an essentially different kind from those of other field theories. They are direct consequences of the fact that the light cone itself gets shifted by the non-linearities of the theory. But the light-cone shift is precisely what gives the theory its unique interest, and a special effort should be made to separate the divergences which it generates from other divergences. The latter may well be amenable to standard treatment, if they remain at all.

As for the light-cone-shift it is impossible to foresee what techniques will be necessary in order to recognize it unambiguously and to deal with it. As a pure guess one might imagine that the badly divergent leading terms of a perturbation expansion will prove to be summable to a convergent expression characterizing a "cut-off" frequency which leads to a breakdown of causality in the strict Lorentz-covariant sense and represents the effect of the fluctuations in the light cone from which it originated. It must be confessed, however, that the problem remains shrouded in darkness and that the end of the chapter leaves us only at the beginning of the subject.

Chapel Hill, North Carolina
November 1960

Appendices to Part II

(9.B) Derivation of results used in Section 6.

The computation of the equations for small disturbances starts with the verification of the following variations:

$$\delta(-\dot{z}^2)^{\frac{1}{2}} = -(-\dot{z}^2)^{\frac{1}{2}} v^\alpha v^\beta s_{\alpha\beta} \,, \tag{B.1}$$

$$\delta v^\alpha = v^\alpha v^\beta v^\gamma s_{\beta\gamma} + v^\beta \delta z^\alpha{}_{,\beta} \,, \tag{B.2}$$

$$\delta w_0 = -t^{ab} s_{ab} = -(-\dot{z}^2)^{-\frac{1}{2}} \frac{\partial(z)}{\partial(t,\underset{\sim}{u})} t^{\mu\nu} s_{\mu\nu} \,, \tag{B.3}$$

$$\delta \frac{\partial(z)}{\partial(t,\underset{\sim}{u})} = \frac{\partial(z)}{\partial(t,\underset{\sim}{u})} \delta z^\alpha{}_{,\alpha} \,, \tag{B.4}$$

$$\delta n = -n\,(v^\alpha v^\beta s_{\alpha\beta} + \delta z^\alpha{}_{,\alpha}) \,, \tag{B.5}$$

$$\delta\rho = -\rho(v^\alpha v^\beta s_{\alpha\beta} + \delta z^\alpha{}_{,\alpha}) + n\omega\delta J \,, \tag{B.6}$$

$$\delta\Gamma_{\mu\nu}{}^\sigma = \frac{1}{2} g^{\sigma\tau}(\delta g_{\mu\tau\cdot\nu} + \delta g_{\nu\tau\cdot\mu} - \delta g_{\mu\nu\cdot\tau}) \,, \tag{B.7}$$

$$\delta R_{\mu\nu\sigma}{}^\tau = \delta\Gamma_{\sigma\nu}{}^\tau{}_{\cdot\mu} - \delta\Gamma_{\sigma\mu}{}^\tau{}_{\cdot\nu} \,. \tag{B.8}$$

These expressions are obtained by straightforward extension of the methods begun at the end of Section 5, taking into account the fact that the metric itself, as well as the z's is now subject to variation. Making use of these expressions together with the definition (9.5.76) and the dynamical equations, and taking note of the admonitions expressed in Eqs. (9.6.5), (9.6.6) and (9.6.7), one finds, by direct computation,

$$\begin{aligned} \delta t^{\alpha\beta} = & -t^{\alpha\beta}\delta z^\gamma{}_{,\gamma} + t^{\alpha\gamma}\delta z^\beta{}_{,\gamma} + t^{\beta\gamma}\delta z^\alpha{}_{,\gamma} \\ & + 2\,(v^\alpha v^\gamma t^{\beta\delta} + 2\,v^\beta v^\gamma t^{\alpha\delta} - v^\gamma v^\delta t^{\alpha\beta} - c^{\alpha\beta\gamma\delta}) s_{\gamma\delta} \,, \end{aligned} \tag{B.9}$$

$$\delta T^{\mu\nu} = -(T^{\mu\nu}\delta z^{\sigma})_{\cdot\sigma} + T^{\mu\sigma}\delta z^{\nu}{}_{\cdot\sigma} + T^{\nu\sigma}\delta z^{\mu}{}_{\cdot\sigma}$$

$$+[(\rho + w)v^{\mu}v^{\nu}v^{\sigma}v^{\tau} + 2\, v^{\mu}v^{\sigma}t^{\nu\tau} + 2\, v^{\nu}v^{\sigma}t^{\mu\tau}$$

$$- v^{\mu}v^{\nu}t^{\sigma\tau} - v^{\sigma}v^{\tau}t^{\mu\nu} - c^{\mu\nu\sigma\tau}]\, s_{\sigma\tau} + n\, v^{\mu}v^{\nu}\delta J, \qquad (B.10)$$

$$\delta(T_{\alpha}{}^{\beta}{}_{\cdot\beta}) = - T_{\alpha}{}^{\beta}{}_{\cdot\beta}\delta z^{\gamma}{}_{,\gamma} + T^{\beta\gamma}(2s_{\alpha\beta\cdot\gamma} - s_{\beta\gamma\cdot\alpha})$$

$$+ \{[(\rho + w)v_{\alpha}v^{\beta}v^{\gamma}v^{\delta} + 2\, v_{\alpha}v^{\gamma}t^{\beta\delta} + 2\, v^{\beta}v^{\gamma}t_{\alpha}{}^{\delta}$$

$$- v_{\alpha}v^{\beta}t^{\gamma\delta} - v^{\gamma}v^{\delta}t_{\alpha}{}^{\beta} - c_{\alpha}{}^{\beta\gamma\delta}]s_{\gamma\delta} + n\, v_{\alpha}v^{\beta}\delta J\}_{\cdot\beta}, \qquad (B.11)$$

$$\delta R_{\mu\nu} = \frac{1}{2}\, g^{\sigma\tau}(\delta g_{\mu\nu\cdot\sigma\tau} + \delta g_{\sigma\tau\cdot\mu\nu} - \delta g_{\mu\sigma\cdot\nu\tau} - \delta g_{\nu\sigma\cdot\mu\tau}), \qquad (B.12)$$

$$\delta G^{\mu\nu} = \frac{1}{2}\, g^{\frac{1}{2}}\, (g^{\mu\sigma}g^{\nu\tau} - \frac{1}{2}\, g^{\mu\nu}g^{\sigma\tau})g^{\rho\lambda}(\delta g_{\sigma\tau\cdot\rho\lambda} + \delta g_{\rho\lambda\cdot\sigma\tau} - \delta g_{\sigma\rho\cdot\tau\lambda} - \delta g_{\tau\lambda\cdot\sigma\rho})$$

$$+ \frac{1}{2}\, g^{\frac{1}{2}}(g^{\mu\nu}R^{\sigma\tau} + g^{\sigma\tau}R^{\mu\nu} - 2\, g^{\mu\sigma}R^{\nu\tau} - 2\, g^{\nu\sigma}R^{\mu\tau} + g^{\mu\sigma}g^{\nu\tau}R$$

$$- \frac{1}{2}\, g^{\mu\nu}g^{\sigma\tau}R)\delta g_{\sigma\tau}\,. \qquad (B.13)$$

Use of these expressions in the variation of the dynamical equations (9.6.9), (9.6.10), (9.6.11), (9.6.12) leads at once to Eqs. (9.6.18) through (9.6.21) of the text.

The proof that the Green's functions of Eq. (9.6.29) are consistent with the supplementary conditions (9.6.24), (9.6.25) involves the derivation of some identities satisfied by the wave operators $F_{\alpha J'}$, etc. By a straightforward calculation which makes use of the readily verified identity

$$(t^{\alpha\beta}v^{\gamma})_{\cdot\gamma} \equiv t^{\alpha\gamma}v^{\beta}{}_{\cdot\gamma} + t^{\beta\gamma}v^{\alpha}{}_{\cdot\gamma} + (v^{\alpha}v^{\gamma}t^{\beta\delta} + v^{\beta}v^{\gamma}t^{\alpha\delta} - c^{\alpha\beta\gamma\delta})v_{\gamma\cdot\delta}\,, \qquad (B.14)$$

one finds

$$v^{\alpha}F_{\alpha J'} = -\omega F_{\Theta J'} \qquad (B.15)$$

$$v^{\alpha}F_{\alpha\epsilon'} = -\,[(\rho + w)\, v^{\gamma}v^{\alpha}v^{\beta}\delta_{\alpha\epsilon'\cdot\beta}]_{\cdot\gamma} \,, \tag{B.16}$$

$$v^{\alpha}F_{\alpha}{}^{\epsilon'\zeta'} = -\,\frac{1}{2}\,[(\rho + w)\, v^{\gamma}v^{\alpha}v^{\beta}\delta_{\alpha\beta}{}^{\epsilon'\zeta'}]_{\cdot\gamma} \,. \tag{B.17}$$

From this it follows that

$$-\,(0 \,,\, v^{\gamma'}\delta(z,z') \,,\, 0)$$

$$= \int v^{\alpha}(F_{\alpha J''} \,,\, F_{\alpha\epsilon''} \,,\, F_{\alpha}{}^{\epsilon''\zeta''}) \begin{pmatrix} G^{\pm}{}_{J''\Theta'} & 0 & 0 \\ G^{\pm\epsilon''}{}_{\Theta'} & G^{\pm\epsilon''\gamma'} & G^{\pm\epsilon''}{}_{\gamma'\delta'} \\ G^{\pm}{}_{\epsilon''\zeta''\Theta'} & G^{\pm}{}_{\epsilon''\zeta''}{}^{\gamma'} & G^{\pm}{}_{\epsilon''\zeta''\gamma'\delta'} \end{pmatrix} d^4z''$$

$$= (\omega'\delta(z,z'') \,,\, \mathbf{0} \,,\, 0)$$

$$+ \int F(z,z'')v^{\alpha''}v^{\beta''}\,[(G^{\pm}{}_{\alpha''\Theta'\cdot\beta''} \,,\, G^{\pm}{}_{\alpha''}{}^{\gamma'}{}_{\cdot\beta''} \,,\, G^{\pm}{}_{\alpha''\gamma'\delta'\cdot\beta''})$$

$$+ \frac{1}{2}(G^{\pm}{}_{\alpha''\beta''\Theta'} \,,\, G^{\pm}{}_{\alpha''\beta''}{}^{\gamma'} \,,\, G^{\pm}{}_{\alpha''\beta''\gamma'\delta'})]\; d^4z'' , \tag{B.18}$$

where

$$F(z,z') \equiv -\,[(\rho + w)v^{\alpha}\delta(z,z')]_{\cdot\alpha} \,. \tag{B.19}$$

It may therefore be inferred that

$$v^{\alpha}v^{\beta}(G^{\pm}{}_{\alpha\Theta'\cdot\beta} + \frac{1}{2}\,G^{\pm}{}_{\alpha\beta\Theta'}) = \mathfrak{G}^{\pm}{}_{\omega'} \,, \tag{B.20}$$

$$v^{\alpha}v^{\beta}(G^{\pm}{}_{\alpha}{}^{\gamma'}{}_{\cdot\beta} + \frac{1}{2}\,G^{\pm}{}_{\alpha\beta}{}^{\gamma'}) = \mathfrak{G}^{\pm}{}_{v'} \,, \tag{B.21}$$

$$v_{\alpha}v_{\beta}(G^{\pm}{}_{\alpha\gamma'\delta'\cdot\beta} + \frac{1}{2}\,G^{\pm}{}_{\alpha\beta\gamma'\delta'}) = 0 \,, \tag{B.22}$$

the $\mathcal{G}^{\pm}$ being the Green's functions for the operator (B.19). Hence, finally,

$$v^{\alpha}v^{\beta}s^{\pm}_{\alpha\beta} = \epsilon\int dt'\int d^3\underset{\sim}{u}'\, v^{\alpha}v^{\beta}[(G^{\pm}_{\alpha\Theta'\cdot\beta}\,,\; G^{\pm}{}_{\alpha}{}^{\gamma'}{}_{\cdot\beta}\,,\; G_{\alpha\gamma'\delta'\cdot\beta})$$

$$+\frac{1}{2}(G^{\pm}_{\alpha\beta\Theta'}\,,\; G^{\pm}{}_{\alpha\beta}{}^{\gamma'}\,,\; G^{\pm}_{\alpha\beta\gamma'\delta'})]\begin{pmatrix}\delta A/\delta\Theta' \\ \delta A/\delta z^{\gamma'} \\ \dfrac{(z')}{(t',\underset{\sim}{u}')}\;\dfrac{\delta A}{\delta g_{\gamma'\delta'}}\end{pmatrix}$$

$$= \epsilon\int dt'\int d^3\underset{\sim}{u}'\,\mathcal{G}^{\pm}[\omega'(\delta A/\delta\Theta') + v^{\gamma'}(\delta A/\delta z^{\gamma'})] \qquad (B.23)$$

which vanishes in virtue of the invariance condition (9.6.16), thus confirming the condition (9.6.26).

In order to verify the supplementary condition (9.6.27) one makes use of the additional identities

$$\frac{1}{2}\,g^{\alpha\beta}F_{\beta J'} + F^{\alpha\beta}{}_{J'\cdot\beta} = 0 \quad , \qquad (B.24)$$

$$\frac{1}{2}\,g^{\alpha\beta}F_{\beta\epsilon'} + F^{\alpha\beta}{}_{\epsilon'\cdot\beta} = 0 \quad , \qquad (B.25)$$

$$\frac{1}{2}\,g^{\alpha\beta}F_{\beta}{}^{\epsilon'\zeta'} + F^{\alpha\beta\epsilon'\zeta'}{}_{\cdot\beta} = g^{\frac{1}{2}}g^{\eta\theta}[\tfrac{1}{2}(g^{\alpha\gamma}g^{\beta\delta} - \tfrac{1}{2}\,g^{\alpha\beta}g^{\gamma\delta})\delta_{\gamma\delta}{}^{\epsilon'\zeta'}{}_{\cdot\beta}]_{\cdot\eta\theta}$$

$$- g^{\frac{1}{2}}R^{\alpha}{}_{\eta}[\tfrac{1}{2}(g^{\eta\gamma}g^{\beta\delta} - \tfrac{1}{2}\,g^{\eta\beta}g^{\gamma\delta})\delta_{\gamma\delta}{}^{\epsilon'\zeta'}{}_{\cdot\beta}] \,, \qquad (B.26)$$

which can be obtained by a straightforward calculation making use of the Einstein equations (9.6.12), the commutation laws for covariant differentiation, and the algebraic identities satisfied by the Riemann tensor. Using these relations, together with the easily verified identity

$$\delta^{\alpha\beta}{}_{\gamma'\delta'\cdot\beta} = -\frac{1}{2}(\delta^{\alpha}{}_{\gamma'\cdot\delta'} + \delta^{\alpha}{}_{\delta'\cdot\gamma'}) ; \tag{B.27}$$

we get

$$-\left(0 , \frac{1}{2}\delta^{\alpha\gamma'} , -\frac{1}{2}(\delta^{\alpha}{}_{\gamma'\cdot\delta'} + \delta^{\alpha}{}_{\delta'\cdot\gamma'})\right)$$

$$= \int \left[\frac{1}{2} g^{\alpha\beta}(F_{\beta J''} , F_{\beta\epsilon''} , F_{\beta}{}^{\epsilon''\zeta''}) + (F^{\alpha\beta}{}_{J''} , F^{\alpha\beta}{}_{\epsilon''} , F^{\alpha\beta\epsilon''\zeta''})_{\cdot\beta}\right]$$

$$\times \begin{pmatrix} G^{\pm}{}_{J''\Theta'} & 0 & 0 \\ G^{\pm\epsilon''}{}_{\Theta'} & G^{\pm\epsilon''\gamma'} & G^{\pm\epsilon''}{}_{\gamma'\delta'} \\ G^{\pm}{}_{\epsilon''\zeta''\Theta'} & G^{\pm}{}_{\epsilon''\zeta''}{}^{\gamma'} & G^{\pm}{}_{\epsilon''\zeta''\gamma'\delta'} \end{pmatrix} d^4z''$$

$$= \int F^{\alpha}{}_{\eta''} \frac{1}{2}(g^{\eta''\epsilon''}g^{\beta''\zeta''} - \frac{1}{2}g^{\eta''\beta''}g^{\epsilon''\zeta''})$$

$$\times (G^{\pm}{}_{\epsilon''\zeta''\Theta'} , G^{\pm}{}_{\epsilon''\zeta''}{}^{\gamma'} , G^{\pm}{}_{\epsilon''\zeta''\gamma'\delta'})_{\cdot\beta''} d^4z'' , \tag{B.28}$$

where $F^{\alpha}{}_{\eta''}$ is the wave operator for the Green's functions $\mathfrak{G}^{\pm\alpha}{}_{\beta'}$ of Eq. (9.6.28):

$$F^{\alpha}{}_{\eta''} \equiv g^{\frac{1}{2}}g^{\beta\gamma}\delta^{\alpha}{}_{\eta''\cdot\beta\gamma} - g^{\frac{1}{2}}R^{\alpha}{}_{\beta}\delta^{\beta}{}_{\eta''} . \tag{B.29}$$

From this it follows that

$$(g^{\alpha\gamma}g^{\beta\delta} - \frac{1}{2}g^{\alpha\beta}g^{\gamma\delta})G^{\pm}{}_{\gamma\delta\Theta'\cdot\beta} = 0 , \tag{B.30}$$

$$(g^{\alpha\gamma}g^{\beta\delta} - \frac{1}{2}g^{\alpha\beta}g^{\gamma\delta})\, G^{\pm}{}_{\gamma\delta}{}^{\epsilon'}{}_{\cdot\beta} = \mathfrak{G}^{\pm\alpha\epsilon'} , \tag{B.31}$$

$$(g^{\alpha\gamma}g^{\beta\delta} - \frac{1}{2}g^{\alpha\beta}g^{\gamma\delta})G^{\pm}{}_{\gamma\delta\epsilon'\zeta'\cdot\beta} = -\mathfrak{G}^{\pm\alpha}{}_{\epsilon'\cdot\zeta'} - \mathfrak{G}^{\pm\alpha}{}_{\zeta'\cdot\epsilon'} , \tag{B.32}$$

whence, making use of an integration by parts, we have

$$(g^{\alpha\gamma}g^{\beta\delta} - \tfrac{1}{2}g^{\alpha\beta}g^{\gamma\delta})\delta^{\pm}g_{\gamma\delta\cdot\beta}$$

$$= \epsilon\int dt' \int d^3\underset{\sim}{u}' \, (g^{\alpha\gamma}g^{\beta\delta} - \tfrac{1}{2}g^{\alpha\beta}g^{\gamma\delta})(G^{\pm}{}_{\gamma\delta\Theta'} \; , \; G^{\pm}{}_{\gamma\delta}{}^{\epsilon'} \; , \; G^{\pm}{}_{\gamma\delta\epsilon'\zeta'})_{\cdot\beta}$$

$$\times \begin{pmatrix} \delta A/\delta\Theta' \\ \delta A/\delta z^{\epsilon'} \\ \dfrac{\partial(z')}{\partial(t',\underset{\sim}{u}')} \; \dfrac{\delta A}{\delta g_{\epsilon'\zeta'}} \end{pmatrix}$$

$$= \epsilon\int dt' \int d^3\underset{\sim}{u}' \, G^{\pm\alpha}{}_{\epsilon'}\left[g^{\epsilon'\zeta'}\frac{\delta A}{\delta z^{\zeta'}} + 2\frac{\partial(z')}{\partial(t',\underset{\sim}{u}')}\left(\frac{\delta A}{\delta g_{\epsilon'\zeta'}}\right)_{\cdot\zeta'}\right] \qquad \text{(B.33)}$$

which vanishes in virtue of the invariance condition (9.6.17).

The evaluation of the Green's functions $G^{\pm\alpha}{}_{\Theta'}$, $G^{\pm}{}_{\alpha\beta\Theta'}$ makes use of the property of self-adjointness possessed by the lower right hand corner of the wave-operator matrix in Eq. (9.6.29), a property which may be verified by explicitly computing the integral

$$\int \left[\begin{pmatrix} F_{\alpha\epsilon''} & F_{\alpha}{}^{\gamma'\delta'} \\ F^{\alpha\beta}{}_{\epsilon''} & F^{\alpha\beta\gamma'\delta'} \end{pmatrix} - \begin{pmatrix} F_{\gamma'\alpha} & F^{\gamma'\delta'}{}_{\alpha} \\ F_{\gamma'}{}^{\alpha\beta} & F^{\gamma'\delta'\alpha\beta} \end{pmatrix}\right] \begin{pmatrix} X^{\gamma'} \\ X_{\gamma'\delta'} \end{pmatrix} d^4z'$$

for arbitrary $X^{\gamma'}$ and $X_{\gamma'\delta'}$, and showing that it vanishes. As a result of this self-adjointness the corresponding corner of the Green's-function matrix satisfies the reciprocity relation (see Section 3)

$$\begin{pmatrix} G^{\pm\alpha\gamma'} & G^{\pm\alpha}{}_{\gamma'\delta'} \\ G^{\pm}{}_{\alpha\beta}{}^{\gamma'} & G^{\pm}{}_{\alpha\beta\gamma'\delta'} \end{pmatrix} = \begin{pmatrix} G^{\mp\gamma'\alpha} & G^{\mp}{}_{\gamma'\delta'}{}^{\alpha} \\ G^{\mp}{}_{\gamma'\delta'\alpha\beta} & G^{\mp}{}_{\gamma'\delta'\alpha\beta} \end{pmatrix} \qquad \text{(B.34)}$$

which permits Eqs. (B.21) and (B.22) to be rewritten in the forms

$$v^{\gamma'}v^{\delta'}(G^{\pm\alpha}{}_{\gamma'\cdot\delta'} + \frac{1}{2}G^{\pm\alpha}{}_{\gamma'\delta'}) = v^{\alpha}\mathcal{G}^{\mp}(z',z) \,, \tag{B.35}$$

$$v^{\gamma'}v^{\delta'}(G^{\pm}{}_{\alpha\beta\gamma'\cdot\delta'} + \frac{1}{2}G^{\pm}{}_{\alpha\beta\gamma'\delta'}) = 0 \,. \tag{B.36}$$

The Green's functions (B.34) are precisely those which solve the equation

$$\int \begin{pmatrix} F_{\alpha\epsilon''} & F_{\alpha}{}^{\epsilon''\zeta''} \\ F^{\alpha\beta}{}_{\epsilon''} & F^{\alpha\beta\epsilon''\zeta''} \end{pmatrix} \begin{pmatrix} G^{\pm\epsilon''}{}_{\Theta'} \\ G^{\pm}{}_{\epsilon''\zeta''\Theta'} \end{pmatrix} d^4z'' = -\int \begin{pmatrix} F_{\alpha J''} \\ F^{\alpha\beta}{}_{J''} \end{pmatrix} G^{\pm}{}_{J''\Theta'} \, d^4z'' \,, \tag{B.37}$$

which is derived from Eq. (9.6.29). Therefore, making use of the explicit forms (9.6.32), (9.6.35) and (9.6.47) for the wave operators $F_{\alpha J'}$, $F^{\alpha\beta}{}_{J'}$ and the Green's functions $G^{\pm}{}_{J\Theta}$, and carrying out an integration by parts, one finds

$$\begin{pmatrix} G^{\pm\alpha}{}_{\Theta'} \\ G^{\pm}{}_{c\beta\Theta'} \end{pmatrix} = \int d^4z'' \int d^4z''' \begin{pmatrix} G^{\pm\alpha\gamma''} & G^{\pm\alpha}{}_{\gamma''\delta''} \\ G^{\pm}{}_{\alpha\beta}{}^{\gamma''} & G^{\pm}{}_{\alpha\beta\gamma''\delta''} \end{pmatrix} \begin{pmatrix} F_{\gamma''J'''} \\ F^{\gamma''\delta''}{}_{J'''} \end{pmatrix} G^{\pm}{}_{J'''\Theta'}$$

$$= \int n''\omega'' v^{\gamma''}v^{\delta''} \begin{pmatrix} G^{\pm\alpha}{}_{\gamma''\cdot\delta''} + \frac{1}{2}G^{\pm\alpha}{}_{\gamma''\delta''} \\ G^{\pm}{}_{\alpha\beta\gamma''\cdot\delta''} + \frac{1}{2}G^{\pm}{}_{\alpha\beta\gamma''\delta''} \end{pmatrix} G^{\pm}{}_{J''\Theta'} \, d^4z''$$

$$= \int_{\mp\infty}^{\tau} d\tau'' \int d^3\underset{\sim}{u}'' \, \omega'' \begin{pmatrix} -\, v^{\alpha}\mathcal{G}^{\mp}(z'',z) \\ 0 \end{pmatrix} \delta(\underset{\sim}{u}'',\underset{\sim}{u}') \,. \tag{B.38}$$

Now, the equation which the Green's functions $\mathcal{G}^{\pm}$ satisfy may be written, in terms of the proper time, in the form

$$-\,\partial[(\rho + w_0)\mathcal{G}^{\pm}]/\partial\tau = -\,\delta(\tau - \tau')\delta(\underset{\sim}{u},\underset{\sim}{u}') \tag{B.39}$$

[see Eq. (B.19)], which has the solution

$$\mathfrak{G}^{\pm}(z,z') = \mp (\rho_0 + w_0)^{-1} \theta(\mp(\tau - \tau'))\delta(\underset{\sim}{u},\underset{\sim}{u}'). \tag{B.40}$$

Therefore Eq. (B.38) becomes

$$\begin{pmatrix} G^{\pm\alpha}{}_{\Theta'} \\ G^{\pm}{}_{\alpha\beta\Theta'} \end{pmatrix} = \begin{pmatrix} - v^{\alpha}\omega G^{\pm} \\ 0 \end{pmatrix} \tag{B.41}$$

where the $G^{\pm}$ are the functions defined in Eq.(9.6.52) of the text.

(9.C) The stressless medium in a fixed metric.

For a stressless medium we have $w_0 = 0$, $\dot{\rho}_0 = 0$, $v^\nu v^\mu{}_{.\nu} = 0$, $T^{\mu\nu} = \rho v^\mu v^\nu$. We retain the framework of clocks in order to have an intrinsic proper time in terms of which to define covariant Poisson brackets. However, we assume the passage to the intrinsic proper time has already been made, and we ignore the clock variables. Only the Green's functions $G^{\pm\alpha\beta'}$, $G^\pm$ remain, satisfying the equations

$$- \rho v^\gamma v^\delta (G^{\pm\alpha\beta'}{}_{.\gamma\delta} - R^\alpha{}_{\gamma\epsilon\delta} G^{\pm\epsilon\beta'}) = - \delta^{\alpha\beta'} , \tag{C.1}$$

$$- (\rho v^\alpha v^\beta G^\pm{}_{.\beta})_{.\alpha} = - \delta(z, z'). \tag{C.2}$$

The solutions of Eq. (C.2) are

$$G^\pm(z, z') = \mp \rho_0{}^{-1} \theta(\mp(\tau - \tau'))(\tau - \tau')\delta(\underset{\sim}{u}, \underset{\sim}{u}'), \tag{C.3}$$

yielding

$$G(z,z') = - \rho_0{}^{-1}(\tau - \tau') \delta(\underset{\sim}{u}, \underset{\sim}{u}') . \tag{C.4}$$

In order to solve Eq. (C.1) we introduce the two-point function $\sigma(z, z')$ which equals one half the square of the distance along the geodesic between z and z'. Its defining equations are

$$\left.\begin{aligned} &\tfrac{1}{2} \sigma_{.\alpha} \sigma_.{}^\alpha = \sigma , \\ &\lim_{z' \to z} \sigma = 0 , \quad \lim_{z' \to z} \sigma_{.\alpha} = 0 ; \quad \lim_{z' \to z} \sigma_{.\alpha\beta} = g_{\alpha\beta} ; \end{aligned}\right\} \tag{C.5}$$

from which we obtain

$$\sigma_{.\gamma}\sigma_.{}^\gamma{}_\beta - \sigma_{.\beta} = 0 , \qquad \sigma_{.\alpha\gamma}\sigma_.{}^\gamma - \sigma_{.\alpha} = 0 , \tag{C.6}$$

$$\sigma_{\cdot\ \gamma}^{\ \alpha}\sigma_{\cdot\ \beta}^{\ \gamma} + \sigma_{\cdot\ \beta}^{\ \gamma\alpha} - \sigma_{\cdot\ \beta}^{\ \alpha} = 0 \ , \tag{C.7}$$

$$\sigma_{\cdot\alpha\gamma\beta'}\sigma_{\cdot}^{\ \gamma} + \sigma_{\cdot\alpha\gamma}\sigma_{\cdot\ \beta'}^{\ \gamma} - \sigma_{\cdot\alpha\beta'} = 0 \ . \tag{C.8}$$

It is convenient also to introduce the two-point function

$$D_{\alpha\beta'} \equiv -\sigma_{\cdot\alpha\beta'} \ , \qquad \lim_{z' \to z} D_{\alpha\beta'} = g_{\alpha\beta} \ , \tag{C.9}$$

and its inverse $D^{-1\alpha\beta'}$, which exists at least when z and z' are sufficiently close together:

$$D_{\alpha\gamma'}D^{-1\beta\gamma'} = \delta_{\alpha}^{\ \beta} \ , \qquad D_{\gamma\alpha'}D^{-1\gamma\beta'} = \delta_{\alpha}^{\ \beta} \ . \tag{C.10}$$

We note that indices induced by covariant differentiation commute when they refer to different points. From Eq. (C.8) we therefore infer

$$\sigma_{\cdot}^{\ \gamma}D_{\alpha\beta';\gamma} + \sigma_{\cdot\alpha}^{\ \ \gamma}D_{\gamma\beta'} - D_{\alpha\beta'} = 0 \ , \tag{C.11}$$

which, together with the law of differentiation of inverse matrices, yields

$$\sigma_{\cdot}^{\ \gamma}D^{-1\alpha\beta'}{}_{;\gamma} - \sigma_{\cdot\ \gamma}^{\ \alpha}D^{-1\gamma\beta'} + D^{-1\alpha\beta'} = 0 \ . \tag{C.12}$$

Since the particle world-lines of the stressless medium are geodesics we here have

$$\sigma_{\cdot\alpha}\delta(\underset{\sim}{u}, \underset{\sim}{u}') = v_{\alpha}(\tau - \tau')\delta(\underset{\sim}{u}, \underset{\sim}{u}') \ , \quad \sigma_{\cdot\beta'}\delta(\underset{\sim}{u},\underset{\sim}{u}') = -v_{\beta'}(\tau'-\tau)\delta(\underset{\sim}{u},\underset{\sim}{u}') \ , \tag{C.13}$$

and since

$$\sigma_{\cdot\beta'} = \sigma_{\cdot\alpha\beta'}\sigma_{\cdot}^{\ \alpha} = -D_{\alpha\beta'}\sigma_{\cdot}^{\ \alpha} \ , \qquad \sigma_{\cdot}^{\ \alpha} = -D^{-1\alpha\beta'}\sigma_{\cdot\beta'} \ , \tag{C.14}$$

it follows that

$$D^{-1\alpha\beta'} v_{\beta'} \delta(\underset{\sim}{u}, \underset{\sim}{u}') = v^{\alpha} \delta(\underset{\sim}{u}, \underset{\sim}{u}') . \tag{C.15}$$

We now assert that the solutions of Eq. (C.1) are

$$G^{\pm\alpha\beta'} = \mp\rho_0^{-1}\theta(\mp(\tau - \tau'))(\tau - \tau')\delta(\underset{\sim}{u}, \underset{\sim}{u}')D^{-1\alpha\beta'} . \tag{C.16}$$

To prove this we first compute

$$\begin{aligned} v^{\gamma} G^{\pm\alpha\beta'}{}_{.\gamma} &= \mp\rho_0^{-1}\theta(\mp(\tau - \tau'))\delta(\underset{\sim}{u}, \underset{\sim}{u}')(D^{-1\alpha\beta'} + \sigma_{.}{}^{\gamma} D^{-1\alpha\beta'}{}_{.\gamma}) \\ &= \mp \rho_0^{-1}\theta(\mp(\tau - \tau'))\delta(\underset{\sim}{u}, \underset{\sim}{u}')\sigma_{.}{}^{\alpha}{}_{\gamma} D^{-1\gamma\beta'} , \end{aligned} \tag{C.17}$$

in which Eqs. (C.12) and (C.13) are used. Then, making use consecutively of Eqs. (C.12), (C.5), (C.9), (C.7) and the commutation law for covariant differentiation, we get

$$\begin{aligned} v^{\gamma} v^{\delta} G^{\pm\alpha\beta'}{}_{.\gamma\delta} &= \rho^{-1}\delta(z, z')\sigma_{.}{}^{\alpha}{}_{\gamma} D^{-1\gamma\beta'} \\ &\quad \mp \rho_0^{-1}\, \theta(\mp(\tau - \tau'))\delta(\underset{\sim}{u}, \underset{\sim}{u}')(\tau - \tau')^{-1} \\ &\quad \times (\sigma_{.\gamma}\sigma_{.}{}^{\alpha}{}_{\delta}{}^{\gamma} + \sigma_{.}{}^{\alpha}{}_{\gamma}\sigma_{.}{}^{\gamma}{}_{\delta} - \sigma_{.}{}^{\alpha}{}_{\delta})D^{-1\delta\beta'} \\ &= \rho^{-1}\, \delta^{\alpha\beta'} \\ &\quad \mp \rho_0^{-1}\, \theta(\mp(\tau - \tau'))\delta(\underset{\sim}{u}, \underset{\sim}{u}')(\tau - \tau')^{-1} R^{\alpha\gamma}{}_{\delta}{}^{\epsilon}\sigma_{.\gamma}\sigma_{.\epsilon} D^{-1\delta\beta'} \\ &= \rho^{-1}\delta^{\alpha\beta'} + v^{\gamma} v^{\delta} R^{\alpha}{}_{\gamma\epsilon\delta} G^{\pm\epsilon\beta'} , \end{aligned} \tag{C.18}$$

from which Eq. (C.1) follows. We therefore have

$$G^{\alpha\beta'} = - \rho_0^{-1}(\tau - \tau')\delta(\underset{\sim}{u}, \underset{\sim}{u}')D^{-1\alpha\beta'} , \tag{C.19}$$

and from Eq. (9.6.53) of the text we obtain the Poisson bracket

$$(z^{\alpha}, z^{\beta'}) = G^{\alpha\beta'} + v^{\alpha}Gv^{\beta'}$$

$$= -\rho_0^{-1}(\tau - \tau')\delta(\underset{\sim}{u}, \underset{\sim}{u}')P^{\alpha\beta'} , \tag{C.20}$$

$$P^{\alpha\beta'} \equiv D^{-1\alpha\beta'} + v^{\alpha}v^{\beta'} , \tag{C.21}$$

which may be compared with Eq. (9.4.50) for the single relativistic particle in flat space-time.

From Eq. (C.15) it follows that

$$v_{\alpha}P^{\alpha\beta'}\delta(\underset{\sim}{u}, \underset{\sim}{u}') = 0 , \qquad P^{\alpha\beta'}v_{\beta'}\delta(\underset{\sim}{u}, \underset{\sim}{u}') = 0 . \tag{C.22}$$

This relation, together with Eq. (9.5.80), enables us to show the consistency of the Poisson bracket (C.20) with the restriction $v_{\alpha}v^{\alpha} = -1$. We have

$$\left((-v_{\alpha}v^{\alpha})^{\frac{1}{2}} , z^{\beta'} \right) = - v_{\alpha}v^{\gamma}(z^{\alpha}, z^{\beta'})_{.\gamma}$$

$$= - v^{\gamma}[v_{\alpha}(z^{\alpha}, z^{\beta'})]_{.\gamma} = 0 . \tag{C.23}$$

The fact that it is the function $D^{-1\alpha\beta'}$ which appears in the Poisson bracket (C.20) may be understood in terms of the disturbance in the momentum of a constituent particle, and hence in the direction of its world line, which results from a measurement of its position. For $D^{-1\alpha\beta'}$ may be recognized as the matrix representing the transformation from the variables z^{α}, $z^{\beta'}$ which specify the geodesic between z and z' by means of its end points, to the variables $z^{\beta'}$, $\sigma_{.\beta'}$ which specify it by means of <u>one</u> of its end points and the tangent vector at that point. If the tangent vector $\sigma_{.\beta'}$ is varied, the resulting variation in z^{α} is *

$$\delta z^{\alpha} = - D^{-1\alpha\beta'}\delta\sigma_{.\beta'} . \tag{C.24}$$

* The matrix $D^{-1\alpha\beta'}$ evidently becomes singular on the caustic surfaces where the geodesics emanating from a given point begin to cross.

(9.D) The spinning particle in a gravitational field.

We consider first the Thomas precession. For this purpose it is convenient to introduce the three unit vectors n_i^{α} of Section 5 [see Eqs.(9.5.64)] along the world line of the particle. The condition that the local Cartesian frame defined by these vectors propagate in as parallel a fashion as possible is

$$n_i^{\alpha}\, \dot{n}_{j\alpha} = 0 \quad , \tag{D.1}$$

the dot denoting the covariant proper time derivative. Taking the covariant proper time derivative of the first of Eqs. (9.5.64) we also have

$$\dot{n}_{i\alpha}\, \dot{z}^{\alpha} + n_{i\alpha}\, \ddot{z}^{\alpha} = 0 \quad , \tag{D.2}$$

which, together with (D.1), implies

$$\dot{n}_i^{\alpha} = \dot{z}^{\alpha} n_{i\beta}\, \ddot{z}^{\beta} \quad . \tag{D.3}$$

The spin angular momentum tensor is first defined in the local Cartesian system. Denoting it by $\Sigma_{ij}(= - \Sigma_{ji})$ in this system, we may express its components in an arbitrary system in the form

$$\Sigma^{\alpha\beta} = n_i^{\alpha} n_j^{\beta} \Sigma_{ij} \quad , \tag{D.4}$$

which automatically satisfies Eqs. (9.7.34). The Thomas precession is obtained by requiring that the spin angular momentum tensor be constant in the local Cartesian system. That is,

$$\dot{\Sigma}_{ij} = 0 \; . \tag{D.5}$$

Combining this with Eq. (D.3) we obtain Eq. (9.7.36) of the text.

The definition of stress-energy density which leads to the ponderomotive equation (9.7.35) as well as to the law of Thomas precession is

$$T^{\mu\nu} \equiv \int [(m\dot{z}^{\alpha}\dot{z}^{\beta} - \dot{z}^{\alpha}\dot{\Sigma}^{\beta}{}_{\gamma}\dot{z}^{\gamma})\delta^{\mu\nu}{}_{\alpha\beta} - \dot{z}^{\alpha}\Sigma^{\beta\gamma}\delta^{\mu\nu}{}_{\alpha\beta\cdot\gamma}]\, d\tau\,, \tag{D.6}$$

where m is the particle rest mass and $\delta^{\mu\nu}{}_{\alpha\beta}$ is the delta function defined by Eq. (9.6.39) with z' replaced by x. Making use of the identity

$$\delta^{\mu\nu}{}_{\alpha\beta\cdot\nu} \equiv -\frac{1}{2}(\delta^{\mu}{}_{\alpha\cdot\beta} + \delta^{\mu}{}_{\beta\cdot\alpha}) \tag{D.7}$$

as well as the laws for interchanging the order of covariant differentiation, we may write the energy-momentum conservation law in the form

$$\begin{aligned}
0 &= T^{\mu\nu}{}_{\cdot\nu} \\
&= \int [-\frac{1}{2}(m\dot{z}^{\alpha}\dot{z}^{\beta} - \dot{z}^{\alpha}\dot{\Sigma}^{\beta}{}_{\gamma}\dot{z}^{\gamma})(\delta^{\mu}{}_{\alpha\cdot\beta} + \delta^{\mu}{}_{\beta\cdot\alpha}) + \frac{1}{2}\dot{z}^{\alpha}\Sigma^{\beta\gamma}(\delta^{\mu}{}_{\alpha\cdot\beta} + \delta^{\mu}{}_{\beta\cdot\alpha})_{\cdot\gamma}]\, d\tau \\
&= \int [(-m\ddot{z}^{\alpha} + \dot{\Sigma}^{\alpha}{}_{\gamma}\dot{z}^{\gamma})\delta^{\mu}{}_{\alpha} - \frac{1}{2}(\dot{z}^{\beta}\dot{\Sigma}^{\alpha}{}_{\gamma} - \dot{z}^{\alpha}\dot{\Sigma}^{\beta}{}_{\gamma})\dot{z}^{\gamma}\delta^{\mu}{}_{\alpha\cdot\beta} + \frac{1}{4}\dot{z}^{\alpha}\Sigma^{\beta\gamma}R_{\beta\gamma\alpha}{}^{\epsilon}\delta^{\mu}{}_{\alpha} \\
&\qquad + \frac{1}{2}\Sigma^{\beta\gamma}\delta(\delta^{\mu}{}_{\beta\cdot\gamma})/\delta\tau + \frac{1}{2}\dot{z}^{\alpha}\Sigma^{\beta\gamma}R_{\alpha\gamma\beta}{}^{\epsilon}\delta^{\mu}{}_{\epsilon}]\, d\tau \\
&= \int \{(m\ddot{z}^{\alpha} - \ddot{\Sigma}^{\alpha}{}_{\beta}\dot{z}^{\beta} - \dot{\Sigma}^{\alpha}{}_{\beta}\ddot{z}^{\beta} - \frac{1}{2}R^{\alpha}{}_{\beta\gamma\delta}\dot{z}^{\beta}\Sigma^{\gamma\delta})\delta^{\mu}{}_{\alpha} \\
&\qquad - \frac{1}{4}[\dot{\Sigma}^{\alpha\beta} + (\dot{z}^{\beta}\dot{\Sigma}^{\alpha}{}_{\gamma} - \dot{z}^{\alpha}\dot{\Sigma}^{\beta}{}_{\gamma})\dot{z}^{\gamma}](\delta^{\mu}{}_{\alpha\cdot\beta} - \delta^{\mu}{}_{\beta\cdot\alpha})\}\, d\tau\,.
\end{aligned} \tag{D.8}$$

The coefficients of the delta function $\delta^{\mu}{}_{\alpha}$ and of the curl $\delta^{\mu}{}_{\alpha\cdot\beta} - \delta^{\mu}{}_{\beta\cdot\alpha}$ must vanish separately in the integrand, and hence we have

$$m\ddot{z}^{\alpha} = \ddot{\Sigma}^{\alpha}{}_{\beta}\dot{z}^{\beta} + \dot{\Sigma}^{\alpha}{}_{\beta}\ddot{z}^{\beta} + \frac{1}{2}R^{\alpha}{}_{\beta\gamma\delta}\dot{z}^{\beta}\Sigma^{\gamma\delta}\,, \tag{D.9}$$

$$\dot{\Sigma}^{\alpha\beta} = -(\dot{z}^{\beta}\dot{\Sigma}^{\alpha}{}_{\gamma} - \dot{z}^{\alpha}\dot{\Sigma}^{\beta}{}_{\gamma})\dot{z}^{\gamma}\,. \tag{D.10}$$

The covariant proper time derivative of the second of Eqs. (9.7.34) allows us immediately to rewrite Eq. (D.10) in the form (9.7.36) of the text. Furthermore, multiplying the latter equation by $\ddot{z}_{\beta}$ and making use of the identity $\dot{z}_{\alpha}\ddot{z}^{\alpha} = 0$ as well as of the antisymmetry of $\Sigma^{\alpha\beta}$, we infer

$$\ddot{\Sigma}^{\alpha}{}_{\beta}\ddot{z}^{\beta} = 0 \ , \tag{D.11}$$

whence Eq. (D.9) reduces to Eq. (9.7.35) of the text.

The stress-energy density for an ensemble of spinning particles labeled by parameters u^a may be obtained from (D.6) by multiplying by the particle number density n_0 in the Lagrangian system and integrating over the u^a :

$$\begin{aligned} T^{\mu\nu} &= \int d\tau \int d^3\underset{\sim}{u}\, n_0 [(m\dot{z}^{\alpha}\dot{z}^{\beta} - \dot{z}^{\alpha}\dot{\Sigma}^{\beta}{}_{\gamma}\dot{z}^{\gamma})\delta^{\mu\nu}{}_{\alpha\beta} - \dot{z}^{\alpha}\Sigma^{\beta\gamma}\delta^{\mu\nu}{}_{\alpha\beta\cdot\gamma}] \\ &= \rho v^{\mu}v^{\nu} - \frac{1}{2}(v^{\mu}\sigma^{\nu}{}_{\sigma\cdot\tau} + v^{\nu}\sigma^{\mu}{}_{\sigma\cdot\tau})v^{\sigma}v^{\tau} + \frac{1}{2}(v^{\mu}\sigma^{\nu\sigma} + v^{\nu}\sigma^{\mu\sigma})_{\cdot\sigma} \ , \end{aligned} \tag{D.12}$$

where ρ is the proper rest mass density and $\sigma^{\mu\nu}$ is the spin density:

$$\rho \equiv nm \ , \tag{D.13}$$

$$\sigma^{\mu\nu} \equiv n\Sigma^{\mu\nu} \ , \qquad \sigma^{\mu\nu}v_{\nu} = 0 \ . \tag{D.14}$$

Equation (D.12) may be used to verify the identification of $\Sigma^{\alpha\beta}$ with the spin angular momentum tensor of a constituent particle. In the case of uniform motion in a flat space-time the momentum density of the ensemble in a Cartesian rest frame is entirely due to the spin and is given by

$$T_a{}^{(0)} = \frac{1}{2}\sigma_{ab,b} \ . \tag{D.15}$$

According to the conventional definition we then have, for the total angular momentum,

$$\begin{aligned} \Sigma_a^{tot} &= \epsilon_{abc}\int x_b T_c{}^{(0)} d^3\underset{\sim}{x} = \frac{1}{2}\epsilon_{abc}\int x_b\sigma_{cd,d}\, d^3\underset{\sim}{x} \\ &= \int \sigma_a\, d^3\underset{\sim}{x} = \int n_0\, \Sigma_a\, d^3\underset{\sim}{x} \ , \end{aligned} \tag{D.16}$$

where

$$\sigma_a \equiv \frac{1}{2} \epsilon_{abc} \sigma_{bc} \, , \qquad \sigma_{ab} \equiv \epsilon_{abc} \sigma_c \, , \qquad \text{(D.17)}$$

$$\Sigma_a \equiv \frac{1}{2} \epsilon_{abc} \Sigma_{bc} \, , \qquad \Sigma_{ab} \equiv \epsilon_{abc} \Sigma_c \, . \qquad \text{(D.18)}$$

Bibliography

J. L. Anderson, Revista Mexicana de Fisica, 3, 176 (1954).

________, Phys. Rev. 110, 1197 (1958).

________, Phys. Rev. 111, 965 (1958).

________, Phys. Rev. 114, 1182 (1959).

R. Arnowitt and S. Deser, Phys. Rev. 113, 745 (1959).

________, S. Deser and C. W. Misner, Phys. Rev. 116, 1322 (1959).

____________________________, Phys. Rev. 117, 1595 (1960).

____________________________, Nuov. Cim. 15, 487 (1960).

____________________________, Phys. Rev. 118, ... (1960).

____________________________, Journ. Math. Phys. 1, ... (1960).

____________________________, Annals of Phys. 11, ... (1960).

L. Bel, Report of the Colloque Internationale sur les Théories Relativistes de la Gravitation, Abbaye de Royaumont (1959).

F. J. Belinfante and J. C. Swihart, Annals of Phys. 1, 168 (1957).

P. G. Bergmann, Helv. Phys. Acta Suppl. 4, 79 (1956).

________, Goldberg, Janis and Newman, Phys. Rev. 103, 807 (1956).

________ and A. I. Janis, Phys. Rev. 111, 1191 (1958).

________ and A. Komar, Phys. Rev. Letters 4, 432 (1960).

G. D. Birkhoff, Proc. Nat. Acd. Sci. 30, 54 (1944).

N. Bohr and L. Rosenfeld, Det Kgl. Danske Vidensk. Selskeb, Mat.-fys. Med. 12, No. 8 (1933).

____________________, Phys. Rev. 78, 794 (1950).

R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves, Interscience Publishers, Inc., New York (1948).

S. Deser, Revs. Modern Phys. 29, 417 (1957).

B. S. DeWitt and R. W. Brehme, Annals of Phys. 9, 220 (1960).

________, Journ. Math. Phys. (1961). (Exact reference to be supplied in proof.)

P. A. M. Dirac, Can. Journ. Math. 2, 129 (1950).

________, Proc. Roy. Soc. (London) A246, 333 (1958).

________, Phys. Rev. 114, 924 (1959).

H. Everett ,III, Revs. Modern Phys. 29, 454 (1957).

R. P. Feynman, Report of the Conference on the Role of Gravitation in Physics, Chapel Hill, North Carolina (1957).

Y. Fourès-Bruhat, J. Rational Mech. Anal. 4, 951 (1956).

J. Géhéniau and R. Debever, Helv. Phys. Acta Suppl. 4, 101 (1956).

S. N. Gupta, Proc. Phys. Soc. (London) A65, 161, 608 (1952).

________, Revs. Modern Phys. 29, 334 (1957).

J. Hadamard, Lectures on Cauchy's Problem in Linear Partial Differential Equations, Yale University Press, New Haven (1923).

W. Heisenberg, Essay in Niels Bohr and the Development of Physics, Pergammon Press, London (1955).

W. Heitler, The Quantum Theory of Radiation, Third Edition, Oxford (1954).

G. Herglotz, Ann. Phys. Lpz. 36, 493 (1911).

O. Klein, Kosmos 32, 33 (1954).

________, Essay in Niels Bohr and the Development of Physics, Pergammon Press, London (1955).

________, Helv. Phys. Acta Suppl. 4, 58 (1956).

________, Nuov. Cim. Suppl. 6, 344 (1957).

A. Komar, Phys. Rev. 111,1182 (1958).

L. Landau and R. Peierls. Zs. f. Phys. 69, 56 (1931).

________, Essay in Niels Bohr and the Development of Physics, Pergammon Press, London (1955).

B. E. Laurent, Arkiv Fysik, 16, 237, 247, 279 (1959).

C. W. Misner, Revs. Modern Phys. 29, 497 (1957).

__________ and J. A. Wheeler, Annals of Phys. 2, 525 (1957).

C. Møller, Det Kgl. Danske Vidensk. Selskab., Mat.-fys. Med. 30, No. 10 (1955).

M. Moshinsky, Phys. Rev. 80, 514 (1950).

E. Noether, Gott. Nachr. (1918) p. 211.

A. Papapetrou, Proc. Roy. Soc. (London) A209, 248 (1951).

W. Pauli, Relativitätstheorie, article in Encyklopädie der mathematischen Wissenschaften, Vol. V19, B. G. Teubner, Leipzig (1921).

________, Helv. Phys. Acta Suppl. 4, 69 (1956).

R. E. Peierls, Proc. Roy. Soc. (London) A214, 143 (1952).

F. A. E. Pirani, Acta Phys. Polon. 15, 389 (1956).

________, Phys. Rev. 105, 1089 (1957).

I. Robinson, Report of the Colloque Internationale sur les Théories Relativistes de la Gravitation, Abbaye de Royaumont (1959).

L. Rosenfeld, Zs. f. Phys. 65, 589 (1930).

________, Report of the Conference on the Role of Gravitation in Physics, Chapel Hill, North Carolina (1957).

H. Salecker, Report of the Conference on the Role of Gravitation in Physics, Chapel Hill, North Carolina (1957).

L. I. Schiff, Phys. Rev. Letters 4, 215 (1960).

W. Thirring, Fortschr. der Phys. 7, 79 (1959).

R. Utiyama, Prog. Theor. Phys. Suppl. 9, 19 (1959).

J. Weber, Phys. Rev. 117, 306 (1960).

J. A. Wheeler, Revs. Modern Phys. 29, 463 (1957).

E. P. Wigner, Revs. Modern Phys. 29, 255 (1957).

H. Y. Yeh, Ph. D. Thesis, University of North Carolina, Chapel Hill (1960).

Footnotes

1. See, for example, the introduction to the chapter by Arnowitt, Deser and Misner in this volume.
2. The author hopes Einstein's ghost will forgive him for this remark.
3. Attempts to "evaluate" the integral when the dynamical equations are satisfied generally lead to the trivialities $S = \infty$ or $S = 0$.
4. As is well known from the theory of continuous groups, most of the properties of the group in the large are determined already by the infinitesimal transformations (neighborhood of the unit element). Only the global topology of the group requires separate investigation, but we do not concern ourselves with this here.
5. It is assumed, of course, that the invariance group alone gives rise to the totality of all conditions (9.2.5). That no further conditions can be obtained by taking variational derivatives is assured by the identity (9.2.3).
6. Here we use the word "state" in the classical sense.
7. i.e., the action integral restricted to this time interval. For this comparison it is necessary to choose one of the many otherwise equivalent integrands which does not trivially vanish when the dynamical equations of the apparatus are satisfied.
8. An integration by parts is not permitted, for example, in the integral $\int d^4x'' \int d^4x' \, G^{-ik''} F_{k''j',} \Phi^{j'}$ unless $\Phi^{j'}$ vanishes sufficiently rapidly in the remote past. The integral, in fact, may diverge if the latter criterion is not met. This is merely one aspect of the circumstance that Eqs. (9.3.7 a,b) are of purely formal validity and must be handled with reasonable care.

9. It may be conjectured that this group is, in fact, the group of <u>all</u> mappings of the set of all physically distinct solutions of the dynamical equations into itself. Just as each physically distinct solution is characterized by the values which it gives to all invariants, so should the set of all invariants be able to generate all physically distinct solutions differing infinitesimally from a given one. While the truth of this conjecture can hardly be doubted, the author has not succeeded in proving it for the general case.

10. The group invariance of the Poisson bracket may likewise be checked by direct computation and use of appropriate identities.

11. In nonrelativistic theories it will, of course, be necessary to restrict the formalism to the subgroup of transformations under which space and time coordinates transform independently.

12 e.g., if a displacement δx^μ exists satisfying Killing's equation: $\delta x_{\mu\cdot\nu} + \delta x_{\nu\cdot\mu} = 0$, or, when the trace of the stress-energy density vanishes, the more general equation: $\delta x_{\mu\cdot\nu} + \delta x_{\nu\cdot\mu} = \frac{1}{2} g_{\mu\nu} \delta x^\sigma{}_{\cdot\sigma}$.

13. t is assumed, however, to increase monotonically with proper time.

14. It should not be supposed that the essentially phenomenological description of the clock fails to lead to a well defined quantum theory --- quite the contrary. Introducing eigenvectors $|p'\rangle$ of the momenta $p_\mu = m\dot{x}_\mu$, and noting that the latter are constrained by the condition $p^2 = -m^2$, one sees that the simplest covariant normalization-completeness condition, which can be written for the former is

$$\int d\rho(m') \int_{p'^0>0} d^4p' \; |p'\rangle \delta(p'^2 + m'^2) \langle p'| = 1 \quad ,$$

where $\rho(m')$ is a monotonically increasing function characterizing the mass spectrum of the clock. Noting, further, that Eqs. (9.4.50) and (9.4.53) imply

$$[x^\mu, p_\nu] = [x^\mu, m\dot{x}_\nu] = i(-\dot{x}^\mu \dot{x}_\nu + p^\mu{}_\nu) = i\,\delta^\mu{}_\nu$$

and

$$\frac{\partial}{\partial \tau'} \langle x'(\tau')| = i\langle x'(\tau')|m \quad ,$$

where the $|x'(\tau')\rangle$ are eigenvectors of the $x^\mu(\tau)$, one may conveniently take

$$\langle x'(\tau')|p'\rangle = (2\pi)^{-\frac{3}{2}} e^{i(p'x' + m'\tau')}$$

$(p'x' \equiv p'_\mu x'^\mu)$, and hence

$$\langle x'(\tau')|x''(\tau'')\rangle = i\int \Delta^{(+)}(x' - x'';m')e^{im'(\tau' - \tau'')}\, d\rho(m') \ ,$$

where $\Delta^{(+)}(x' - x'';m')$ is the positive energy component of the familiar propagation function for a relativistic particle of mass m'.

15. Young's modulus, the bulk modulus and Poisson's ratio are given respectively by

$$Y = 2\mu(1 + \sigma) = 3k(1 - 2\sigma), \quad k = \lambda + \frac{2}{3}\mu \ , \quad \sigma = \frac{1}{2}\lambda/(\lambda + \mu).$$

(See, for example, American Institute of Physics Handbook, McGraw-Hill, New York, 1957, p. 2 - 10.)

16. The positive and negative frequency components are uniquely defined by the equations

$$\delta x_a = \delta x_a^{(+)} + \delta x_a^{(-)} \ , \quad \delta x_a^{(-)} = \delta x_a^{(+)\dagger} \ , \quad \delta \dot{x}_a^{(+)} = -i\underset{\sim}{\omega}\delta x_a^{(+)} \ ,$$

where $\underset{\sim}{\omega}$ is a positive definite Hermitian differential operator. If the vector $\delta x_a^{(+)}|0\rangle$ does not vanish, then because $\delta \dot{x}_a^{(+)} = -i[\delta x_a^{(+)}, H]$, which implies $H\delta x_a^{(+)}|0\rangle = (E_0 - \underset{\sim}{\omega})\delta x_a^{(+)}|0\rangle$ it follows that $\delta x_a^{(+)}|0\rangle$

must represent a state in which the average energy is lower than that of the ground state. But this is a contradiction since it may be readily shown that the Hamiltonian (9.5.23), with internal energy given by (9.5.39), is positive definite as long as the bulk and shear moduli are positive ($\lambda > -\frac{2}{3}\mu < 0$) and therefore possesses a spectrum bounded from below.

17. This identity is readily verified by considering the minors of the matrix formed by the four vectors n_i^{μ}, $_1 v^{\mu}$ and observing that the determinant of the matrix itself is equal to $g^{-\frac{1}{2}}$. Here it is assumed that the vectors v^{μ}, n_1^{μ}, n_2^{μ}, n_3^{μ} have the same relative orientation as displacements in the x^0, x^1, x^2 x^3 directions respectively and that the permutation symbols are taken with the sign conventions $\epsilon_{123} = 1$, $\epsilon_{0123} = 1$.

18. The symbol $\delta g_{\mu\nu\cdot\sigma\tau}$ is effectively unambiguous. It means, of course, $(\delta g_{\mu\nu})_{\cdot\sigma\tau}$ rather than $\delta(g_{\mu\nu\cdot\sigma\tau})$ since the latter would be trivial. Generally speaking, when parentheses are omited, as in $\delta x_{\mu\cdot\nu}$, $\delta z_{\alpha\cdot\beta}$, $\delta\Gamma_{\mu\nu}{}^{\sigma}{}_{\cdot\tau}$, etc., the covariant derivative is to be understood as performed on the variation, rather than vice versa. The same holds for ordinary derivatives, as in $\delta z^{\alpha}{}_{,\beta}$.

19. It is not hard to show that Eq. (9.6.21) may be rewritten in the form

$$
\begin{aligned}
&g^{\frac{1}{2}}(g^{\mu\sigma}g^{\nu\tau} - \tfrac{1}{2}\, g^{\mu\nu}g^{\sigma\tau})g^{\rho\lambda}(s^{\pm}{}_{\sigma\tau\cdot\rho\lambda} + s^{\pm}{}_{\rho\lambda\cdot\sigma\tau} - s^{\pm}{}_{\sigma\rho\cdot\tau\lambda} - s^{\pm}{}_{\tau\lambda\cdot\sigma\rho}) \\
&\quad + g^{\frac{1}{2}}(g^{\mu\nu}R^{\sigma\tau} + g^{\sigma\tau}R^{\mu\nu} - 2\, g^{\mu\sigma}R^{\nu\tau} - 2\, g^{\nu\sigma}R^{\mu\tau} + g^{\mu\sigma}g^{\nu\tau}R - \tfrac{1}{2}\, g^{\mu\nu}g^{\sigma\tau}R)s^{\pm}{}_{\sigma\tau} \\
&\quad + \tfrac{1}{2}\,[(\rho + w)v^{\mu}v^{\nu}v^{\sigma}v^{\tau} + 2\, v^{\mu}v^{\sigma}t^{\nu\tau} + 2\, v^{\nu}v^{\sigma}t^{\mu\tau} \\
&\qquad - v^{\mu}v^{\nu}t^{\sigma\tau} - v^{\sigma}v^{\tau}t^{\mu\nu} - c^{\mu\nu\sigma\tau}]s^{\pm}{}_{\sigma\tau} + \tfrac{1}{2}\, n\omega v^{\mu}v^{\nu}\delta^{\pm}J \\
&\qquad\qquad = -\,\epsilon\delta A/\delta g_{\mu\nu}\,.
\end{aligned}
$$

20. The delta functions appearing in the definitions of these wave operators are to be taken as densities of weight zero at the point z and unit weight at the point z'. The wave operators themselves are therefore double-densities, having unit weight at both z and z'. The delta functions appearing in the final matrix in Eq. (9.6.29), on the other hand, have unit weight at z and weight zero at z'.

21. One of the drawbacks of the coordinate system provided by the clock framework is that under sustained compressional motion hypersurfaces of constant τ will not remain space-like but will begin to cross the light cone after a period of time of the order of the reciprocal of the velocity gradient, but before infinite compression is reached. This is not a practical difficulty, however, when the elastic medium is used in the ground state, subject only to small oscillations. Oscillations themselves, even when violent, tend to wash out the effect.

22. It is well that such terms in fact appear. It will be noted that the stress-energy density occurs in Eq. (9.7.3) with the opposite sign from what it has in Eqs. (9.3.61), (9.3.62), in analogy with the negative sign on the proper-time-displacement generator m of Eq. (9.4.52). If the first term of Eq. (9.7.3) were to stand alone, energy would then have to be defined as a negative definite quantity. Furthermore, the first term refers only to the proper energy of the elastic medium and cannot describe "energy of the gravitational field," even assuming the concept to be meaningful in the general case.

23. The structure of propagation functions in the presence of a general metric has been described by Hadamard (1923) and by DeWitt and Brehme (1960).

24. The group in this case is Abelian and analogous to the gauge group of electrodynamics.

25. The introduction of the notion of uniform motion right at the start is, of course, antithetical to Mach's Principle. Any approximation scheme, however, is almost bound to be anti-Machian, since Mach's Principle is very closely associated with requirements of self-consistency, which can be tested only after approximate results have themselves been obtained.

26. Equation (9.7.18) has the fully covariant analog

$$g^{\frac{1}{2}}(g^{\rho\lambda}R_{\mu\nu\sigma\tau\cdot\rho\lambda} + \frac{1}{2}R_{\mu}{}^{\rho}R_{\rho\nu\sigma\tau} + \frac{1}{2}R_{\nu}{}^{\rho}R_{\mu\rho\sigma\tau} + \frac{1}{2}R_{\sigma}{}^{\rho}R_{\mu\nu\rho\tau} + \frac{1}{2}R_{\tau}{}^{\rho}R_{\mu\nu\sigma\rho}$$

$$- R_{\mu\nu\rho\lambda}R^{\rho\lambda}{}_{\sigma\tau} - 2R_{\mu\rho\sigma\lambda}R_{\nu}{}^{\rho}{}_{\tau}{}^{\lambda} + 2R_{\mu\rho\tau\lambda}R_{\nu}{}^{\rho}{}_{\sigma}{}^{\lambda})$$

$$= -\frac{1}{4}(U_{\sigma\mu\cdot\tau\nu} + U_{\tau\nu\cdot\sigma\mu} - U_{\tau\mu\cdot\sigma\nu} - U_{\sigma\nu\cdot\tau\mu} + U_{\mu\sigma\cdot\nu\tau} + U_{\nu\tau\cdot\mu\sigma} - U_{\mu\tau\cdot\nu\sigma} - U_{\nu\sigma\cdot\mu\tau}),$$

$$U_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T ,$$

which may be used, among other things, to prove that gravitational wave fronts, represented by discontinuities in the Riemann tensor, propagate along null surfaces (relative to the metric ahead of the front) and have polarization vectors which propagate in a parallel fashion along the null geodesics.

27. For Example: $\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}.$

28. It is assumed that no resonances have built up.

29. We here employ a coordinate system which is extended beyond the initial instant in such a way that g_{11} remains equal to unity. The role of directly defining the coordinate system is thus temporarily withheld from the test body and is only restored to it at the end of the measurement period, when the body regains its previous elastic properties. This, however, does

not mean that the coordinate system thereby becomes any the less "intrinsic," since it is still uniquely determined by a set of initial conditions. Actually, the use of such a system is forced upon us by the fact that the coordinates defined by the test body are rendered temporarily unfit for service by the disturbances which the test body suffers during the measurement process (see below). On the other hand we note that since the weak-field situation is assumed, the commutators of the Riemann tensor in this system are not sensibly different from what they would be in a system defined at all times by an undisturbed body.

30. If they did exist it would be possible to avoid the limitation (9.8.29) on the smallness of measurement domains. On the other hand, difficulties with the stability of the vacuum would then be encountered.

31. Here there is no question of field-theoretical divergences. If one were to calculate explicitly the contribution of the gravitational field to the quantum fluctuations in the invariant strain tensor, for example, the phenomenological cut-off (9.5.56) would have to be used for this contribution just as for the contribution coming from the elastic wave field. This is because the metric at any point in the elastic medium has no meaning other than as an average over a region of volume ℓ^3. The problems of divergence and renormalization must be considered only when the material particles involved are described ab initio in fundamental field theoretical terms. These remarks also hold in the electromagnetic case considered by Bohr and Rosenfeld.

32. The state of the body before and after the field measurement is practically unstrained. A field which is weak enough for quantum effects to be important produces an average strain in a stiff elastic medium of order $R_{crit} L^2 = L^{-1} \lll 1$ or less. The strain Δs_{11}, in contrast, may be much large

33. Its ratio with ΔT is $c^{1111}\Delta s_{11}\Delta s_{11}/\rho_0\Delta s_{11} \ll (\lambda + 2\mu)/\rho_0 = c_s^{\,2} < 1$.

34. The energy density stored in the compensation mechanism may be neglected in comparison with ΔT . To prove this we first note that the sound velocity corresponding to the modulus (9.8.71) is of order

$$c_s = \left(\frac{c_{1111}}{\rho_0}\right)^{\frac{1}{2}} \sim \left(\frac{1}{24}\ \frac{M}{L}\right)^{\frac{1}{2}} \ll 1 .$$

The proof then follows by the inequality of footnote 33.

35. Here again we employ a coordinate system which is detached from the test body during the measurement interval T but which is uniquely specified by initial conditions. (Cf. footnote 29)

36. For a purely group theoretical approach to the quantization of geometry the reader is referred to the papers of Klein (1955) and Laurent (1959).

37. Only the most rudimentary development of the Feynman techniques has so far been achieved in general relativity. For a discussion of the problems involved in this difficult subject see the papers by Misner (1957) and Laurent (1959).